Review problems for the final

1. The parents of a teenager child woke up in the middle of the night and suspected their kid had broken their 11:00 p.m. curfew. The parents noticed that the child had forgotten a cup of coffee outside the front door. Since they were familiar with Newton's law of cooling, they thought they could use it to determine the time the child got home, and so they decided to measure its temperature. The temperature outside was 30°F and had not changed since 6:00 p.m. of the previous day. At 1:00 a.m. the temperature of the coffee was 35°F and at 2:00 a.m. its temperature was 32.5°F. The parents estimated that the temperature of the coffee must have been about 70 degrees (cold enough not to be noticed) when the child got home. At what time did the child get home?

Equation $\frac{dT}{dt} = -k(T-\tau)$.

with $\tau = 30$ and $T(0) = T_0 = 70$.

Data $T(t_0) = 35$ $\quad T(t_0+1) = 32.5$

Unknown t_0 and k

Solution to the equation $T \geq \tau$ and

$\frac{dT}{T-\tau} = -k\,dt$ → separable

Thus

$\ln(T-\tau) = -kt + c_1 \quad , c_1 \in \mathbb{R}$

and if $T(0) = 70$, $c_1 = \ln(40)$. Hence

$\ln(T-\tau) = -kt + \ln(40)$

We have obtained

$$\ln\left(\frac{T-30}{40}\right) = -kt$$

$$\Leftrightarrow \quad kt = \ln\left(\frac{40}{T-30}\right)$$

Plugging our data we have

$$k\,t_0 = \ln\left(\frac{40}{35-30}\right) = \ln(8)$$

$$k\,t_0 + k = \ln\left(\frac{40}{35-32.5}\right) = \ln(16)$$

Thus

(i) $k = \ln(16) - \ln(8) = \ln(2)$

(ii) $t_0 = \dfrac{\ln(8)}{\ln(2)} = \dfrac{\ln(2^3)}{\ln(2)} = 3$

Conclusion

1 am is $t_0 = 3$, thus the teenager came back at $\boxed{10 \text{ pm}}$.

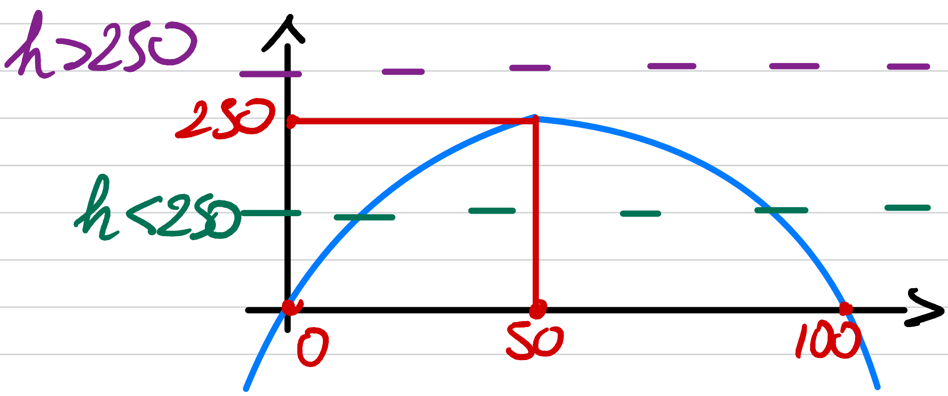
2.. The differential equation

$$\frac{dx}{dt} = \frac{1}{10}x(100-x) - h$$

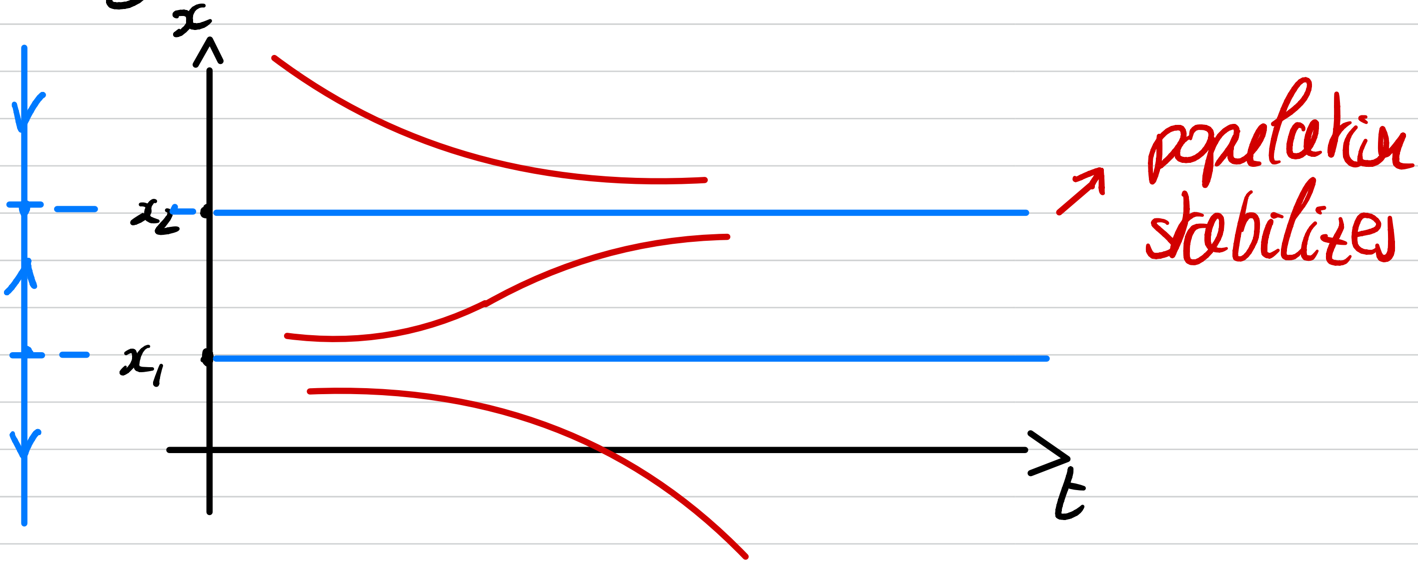
models a logistic population with harvesting at rate h. Determine the range of the values of h such that if the initial population is large enough, the population stabilizes, and does not go extinct for large t.

Let $g(x) = \frac{1}{10} x(100-x)$. Then g admits a maximum

$$g(x^*) = \frac{50}{10} \cdot 50 = 250, \text{ for } x^* = 50$$



Hence if $h < 250$ the phase diagram is of the form



If $h > 250$, the diagram becomes

we get $\boxed{\text{stabilization for } h < 250}$

3. The differential equation

$$\frac{dx}{dt} = \underbrace{\frac{1}{10}x(100-x) - hx}_{f(x)}$$

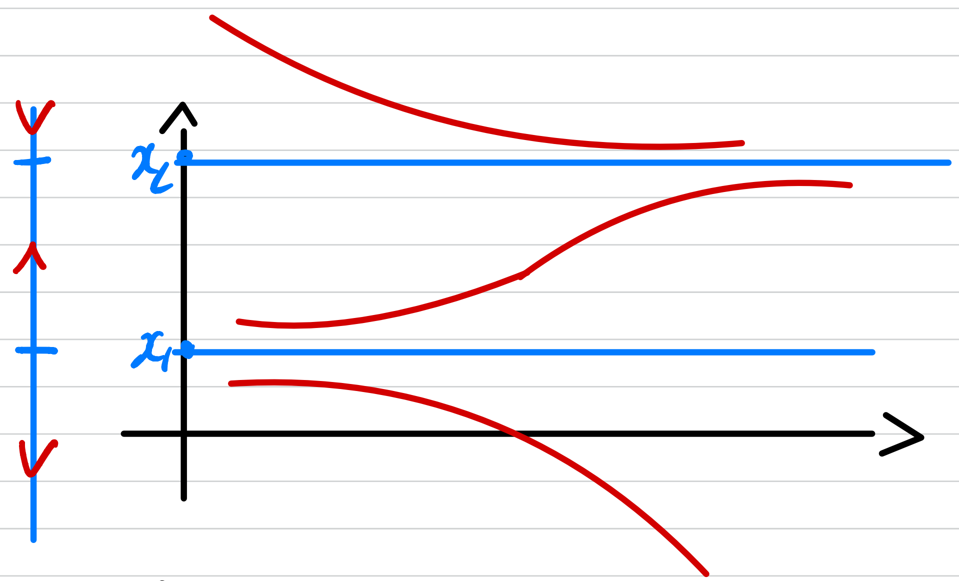
models a logistic population with harvesting at rate hx. Determine the range of h such that the population stabilizes, and does not go extinct for large t, and determine the maximum of hx.

We have $f(x) = \frac{1}{10} x (100 - 10h - x)$

Roots for f: $x_1 = 0$, $x_2 = 10(10-h)$

We get 2 types of diagram

(i) If $h < 10$. Then $x_2 > 0$ and

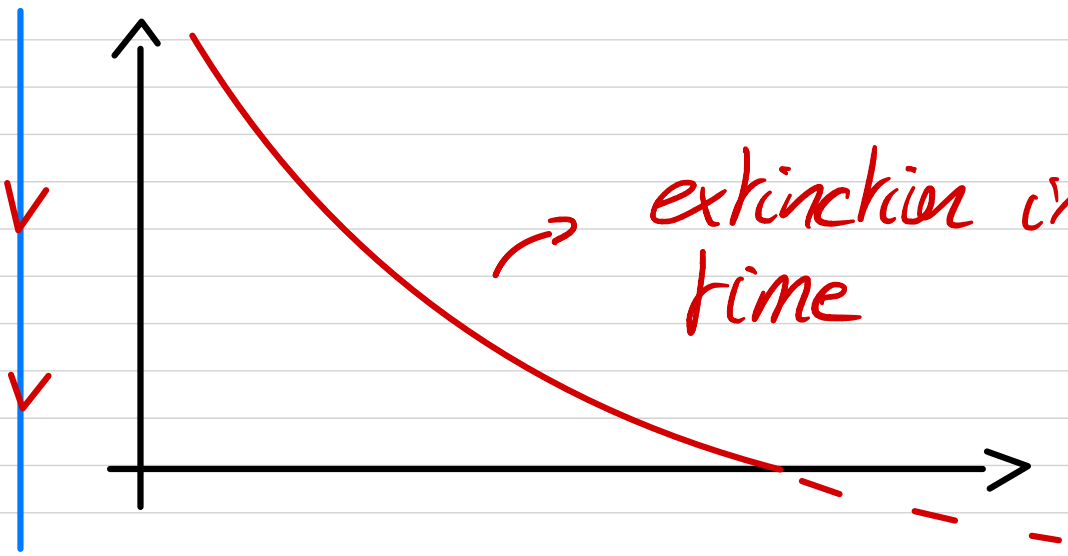


stabilization at $x_2 = 10(10-h)$. Then

$h x_2 = 10h(10-h)$,

maximal at $h = 5$ with max 250

(ii) If $h > 10$. Then $x_2 < 0$ and the diagram becomes



④ Find an implicit solution of the initial value problem:

$$\begin{cases} \overbrace{(6x^2y^2 + 4e^x - 2y\sin 2x)}^{M} + \overbrace{(4x^3y + \cos 2x)}^{N}\dfrac{dy}{dx} = 0, \\ y(0) = 1 \end{cases}$$

Exact equation We compute

$$M_y = 12x^2y - 2\sin(2x)$$

$$N_x = 12x^2y - 2\sin(2x)$$

Since $M_y = N_x$, we have an exact equation.

Computing ϕ Write

$$\phi(x,y) = \int M\,dx$$

$$= \int \left(6x^2y^2 + 4e^x - 2y\sin(2x)\right)dx$$

$$= 2x^3y^2 + 4e^x + y\cos(2x) + h(y)$$

Thus

$$\phi_y = \overbrace{4x^3y + \cos(2x)}^{N} + h'(y)$$

Therefore

$$\Phi_y = N \iff h'(y) = 0$$

Hence the implicit form of the solution is

$$2x^3y^2 + 4e^x + y\cos(2x) = C$$

If $y(0) = 1$ we get

$$C = 4+1 = 5$$

Note: ⓒ is already the only possibility here

Thus the unique solution is

$$\boxed{2x^3y^2 + 4e^x + y\cos(2x) = 5}$$

5 The general solution of $xy' - y = x^2 e^x$ is

Recast the equation as

$$y' - \frac{1}{x} y = x e^x \longrightarrow \text{linear equation}$$

Then

$$\mu = e^{-\int \frac{1}{x}} = \frac{1}{x}$$

Multiplying by $\frac{1}{x}$ we thus get

$$\left(\frac{1}{x} y\right)' = e^x$$

$$\Rightarrow \quad \frac{1}{x} y = e^x + c$$

$$\Rightarrow \quad \boxed{y = x e^x + c x}$$

6 The solution of

$$y' + \frac{y}{x} = \frac{2}{x^2 y}, \qquad x \neq 0$$

is given by

This is a Bernoulli equation. We multiply by y and we get

$$\frac{1}{2} \times 2 y y' + \frac{1}{x} y^2 = \frac{2}{x^2}$$

Next set $u = y^2$. This yields

$$\frac{1}{2} u' + \frac{1}{x} u = \frac{2}{x^2}$$

$$\Leftrightarrow \quad u' + \frac{2}{x} u = \frac{4}{x^2} \longrightarrow \text{linear eq.}$$

Then

$$\mu = e^{\int \frac{2}{x} dx} = x^2$$

Multiplying by μ we obtain

$$(x^2 u)' = 4$$

$$\Rightarrow \quad x^2 u = 4x + c$$

$$\Rightarrow \quad \boxed{x^2 y^2 - 4x = c}$$

7 Find the explicit solution of the initial value problem

$$y' = \frac{xy^2}{x^2+1} \quad , \quad y(0) = 3.$$

Write the eq. under the form

$$\frac{dy}{y^2} = \frac{x}{x^2+1}\, dx \longrightarrow \text{separable}$$

Integrate on both sides

$$-\frac{1}{y} = \frac{1}{2}\ln(x^2+1) + C$$

If $y(0) = 3$, we get $C = -\frac{1}{3}$. Hence

$$-\frac{1}{y} = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}$$

$$\Leftrightarrow \quad y = \frac{1}{1/3 - \frac{1}{2}\ln(x^2+1)}$$

$$\boxed{y = \frac{6}{2 - 3\ln(x^2+1)}}$$

8 Which of the following is the general solution to $y'' + 4y = e^{2t} + 12\sin(2t)$?

(i) Hom. equation. $y'' + 4y = 0$

$P(r) = r^2 + 4$. Roots : $r = \pm 2i$ (simple)

Thus

$$y_c = c_1 \cos(2t) + c_2 \sin(2t)$$

(ii) Particular solution. Of the form

$$y_p = a e^{2t} + t(b\cos(2t) + c\sin(2t)) \quad \times 4$$

$$y_p' = 2a e^{2t} + t(2c\cos(2t) - 2b\sin(2t)) + (b\cos(2t) + c\sin(2t)) \quad \times 0$$

$$y_p'' = 4a e^{2t} + t(-4b\cos(2t) - 4c\sin(2t)) + 4c\cos(2t) + (-4b)\sin(2t) \quad \times 1$$

$$8a e^{2t} + 4c\cos(2t) - 4b\sin(2t)$$

We thus take $a = 1/8$, $c = 0$, $b = -3$.

$$\Rightarrow \boxed{y = c_1\cos(2t) + c_2\sin(2t) + \frac{1}{8}e^{2t} - 3t\cos(2t)}$$

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A particular solution of the equation

$$y^{(4)} - y'' = 2\sin t - 3e^{-t} + 4t$$

(i) Hom. equation 1st polynomial is

$P(r) = r^4 - r^2 = r^2(r^2-1) = r^2(r-1)(r+1)$

Root	Multiplicity
0	2
1	1
-1	1

(ii) Particular solution of the form

$$\boxed{\begin{aligned} y_p = {} & a\cos(t) + b\sin(t) \\ & + cte^{-t} + t^2(dt+e) \end{aligned}}$$

10 Same method

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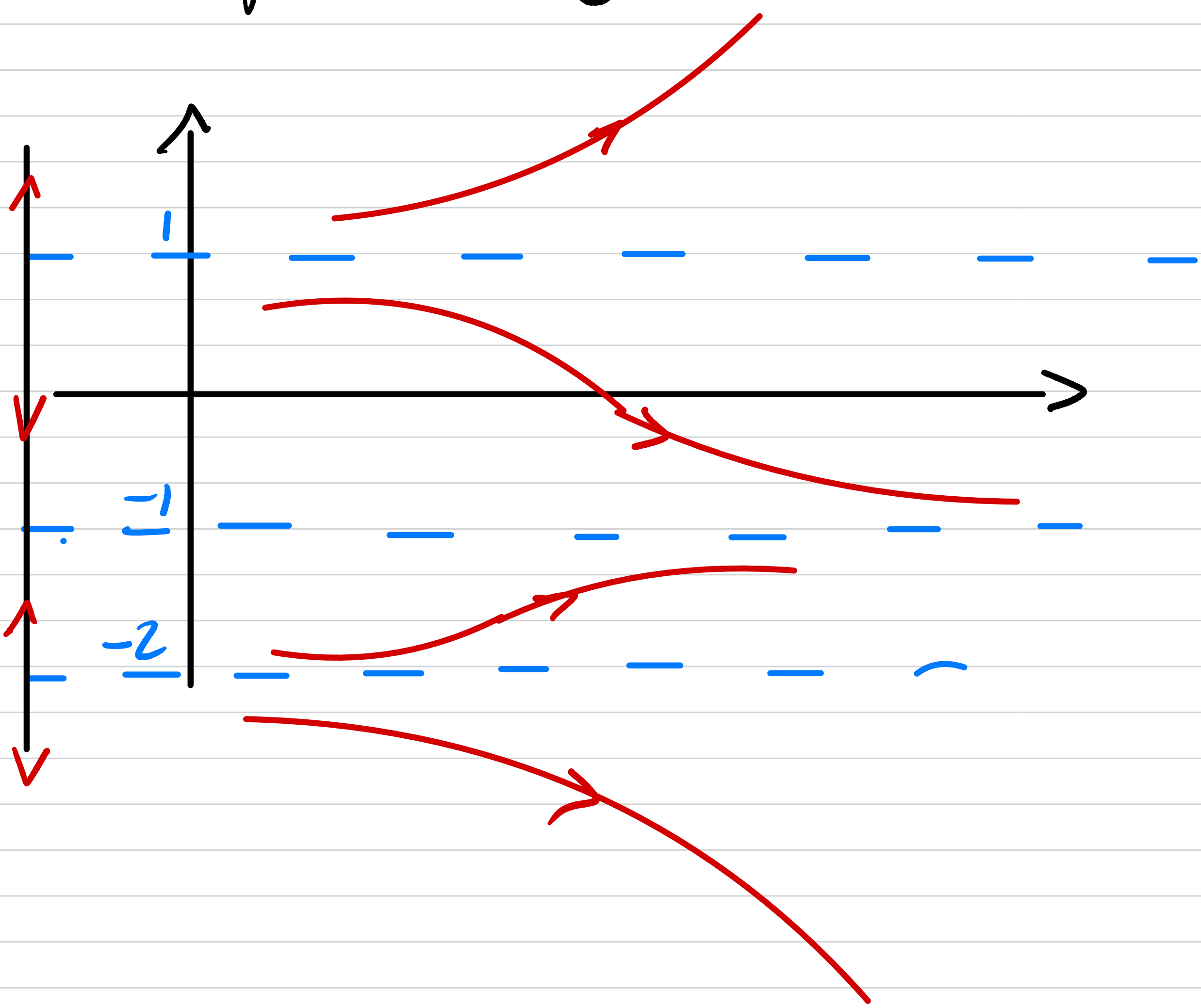
5. How many asymptotically unstable equilibrium solution(s) does the following differential equation have?

$$\underbrace{y' = (y^2+1)(y^2-1)(y+2)}_{f(y)}.$$

We have

$$f(y) = (y^2+1)(y+2)(y+1)(y-1)$$

The phase diagram is



$\boxed{\text{1 and -2 are unstable eq}}$

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Find the solution $y(x)$ to

$$\frac{dy}{dx} = e^{-\frac{y}{x}} + \frac{y}{x},$$
$$y(e) = 0.$$

The rhs is of the form $F(v)$ with

$F(v) = e^{-v} + v$, $v = \frac{y}{x}$ $\longrightarrow$ homogeneous

Thus we set $y = xv$ and we get

$xv' + \cancel{v} = e^{-v} + \cancel{v}$

$\Leftrightarrow e^{v} dv = \frac{1}{x} dx$ $\longrightarrow$ separable

Integrate we get

$e^{v} = \ln|x| + c$

Initial condition For $x = e$, $y = 0$ and thus $v = 0$. This yields

$\overbrace{e^{0}}^{1} = \overbrace{\ln(e)}^{1} + c \Rightarrow c = 0$

Hence

$e^{v} = \ln|x| \Rightarrow \frac{y}{x} = \ln(\ln|x|)$

$\Rightarrow \boxed{y = x \ln(\ln(|x|))}$

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Which of the following is the general solution to $y'' + 4y = e^{2t} + 12\sin(2t)$?

Similar to 8

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Find the general solutions to $y'' + 6y' + 10y = 0$

Auxiliary polynomial

$$P(r) = r^2 + 6r + 10$$
$$= (r+3)^2 + 1$$

Roots

$$r = -3 \pm i$$

General solution

$$y = e^{-3t}(c_1 \cos(t) + c_2 \sin(t))$$

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Find the general solution of the system

$$\mathbf{x}' = \overbrace{\begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}}^{A} \mathbf{x}.$$

Eigenvalues

$$\det(A - \lambda I) = (\lambda+3)(\lambda-1) + 4$$

$$= \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \to -1 \text{ double eig.}$$

Eigenvector

$$A + I = \begin{pmatrix} -2 & -4 \\ 1 & 2 \end{pmatrix}$$

Thus $(A+I)\vec{v} = 0 \iff v_1 = -2v_2$

Take

$$\vec{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Generalized eigenvector

Since $(A+I)^2 = 0$, we take any vector $\vec{v}_2 \not\parallel \vec{v}$, e.g

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Then take

$$\vec{v}_1' = (A+I)\vec{v}_2' = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

General solution of the form

$$X(t) = c_1 e^{-t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$+ c_2 e^{-t} \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

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Find the number of **stable** critical points for the autonomous equation

$$\frac{dx}{dt} = x(x-1)^2(x+3)(x^2-4).$$

Similar to 11