Pb1

Eq: $\underbrace{(y\cos(x) + 2xe^y)}_{M} + \underbrace{(\sin(x) + x^2e^y - 1)}_{N}y' = 0$

$$\left.\begin{array}{l} M_y = \cos(x) + 2xe^y \\ N_x = \cos(x) + 2xe^y \end{array}\right\} \Rightarrow \text{exact equation}$$

Then

(1) $\phi(x,y) = \int M\,dx = y\sin(x) + x^2e^y + h(y)$

and

$\phi_y = N \Leftrightarrow \sin(x) + x^2e^y + h'(y) = \sin(x) + x^2e^y - 1$

Therefore

$h'(y) = -1$ and $h(y) = -y$ $(+c)$

Plugging into (1) we get

$\phi(x,y) = y\sin(x) + x^2e^y - y$

The general solution is thus given by

$y\sin(x) + x^2e^y - y = c$

Pb2 $\quad x^3 y' + 4x^2 y = e^{-x}$ Linear eq.

$\Leftrightarrow \quad y' + \frac{4}{x} y = \frac{e^{-x}}{x^3}$

Integrating factor : I s.t. $\frac{I'}{I} = \frac{4}{x}$
We take $I = x^4$.

Thus eq $\Leftrightarrow \quad (x^4 y)' = x e^{-x}$

$\Leftrightarrow \quad x^4 y = \int x e^{-x} dx = -(x+1)e^{-x} + c$

$\Leftrightarrow \quad y = -\frac{(x+1)}{x^4} e^{-x} + \frac{c}{x^4}$

P63 $y' = \frac{2\cos(x)}{3+2y}$ Separable eq

$\Leftrightarrow (3+2y)\,dy = 2\cos(x)\,dx$

$\Leftrightarrow 3y + y^2 = 2\sin(x) + c$

General solution:

$y^2 + 3y - 2\sin(x) = c$

With initial condition $y(0) = 1$ we find

$c = 4$

Thus the unique solution is given implicitely by

$y^2 + 3y - 2\sin(x) = 4$

Pb 4 $\quad t^2 y' + 2t\, y = y^3$ $\qquad$ Bernoulli

$$\Leftrightarrow \quad t^2 y^{-3} y' + 2t\, y^{-2} = 1$$

Set $u = y^{-2}$. Then $y^{-3} y' = -\frac{1}{2} u'$

The equation becomes

$$-\frac{1}{2} t^2 u' + 2t\,u = 1$$

$$\Leftrightarrow \quad t^2 u' - 4t\,u = -2$$

Pb 5 We have $C_{in} = 2$ $\qquad r_{in} = \gamma$ $\qquad r_{out} = \gamma$

$V(t) = 10$

Eq: $$\frac{dQ}{dt} = r_{in}\, C_{in} - r_{out}\, C_{out}$$

$$= 2\gamma - \frac{Q}{5}$$

Thus

(1) $$Q' + \frac{1}{5}Q = 2\gamma \quad , \quad Q(0) = 5$$

Integrating factor: $I = e^{t/5}$. We get

$$(1) \Leftrightarrow (e^{\frac{t}{5}} Q)' = 2\gamma e^{t/5}$$
$$\Leftrightarrow e^{t/5} Q = 10\gamma e^{t/5} + c$$
$$\Leftrightarrow Q = 10\gamma + c\, e^{-t/5}$$

Moreover

$$Q(0) = 5 \Leftrightarrow 10\gamma + c = 5 \Leftrightarrow c = 5 - 10\gamma$$

We get

$$Q = 10\gamma + (5 - 10\gamma) e^{-t/5}$$

Pb6

$$y' = \frac{2x + (x^3+y^3)^{\frac{1}{3}}}{y} \qquad \text{Homogeneous}$$

Set $v = \frac{y}{x} \iff y = xv$. We get

$$y' = xv' + v,$$

and the equation becomes

$$xv' + v = \frac{2 + (1+v^3)^{\frac{1}{3}}}{v}$$

$$\iff xv' = \frac{2 - v^2 + (1+v^3)^{\frac{1}{2}}}{v}$$

$$\iff \frac{v}{2 - v^2 + (1+v^3)^{\frac{1}{3}}}\, dv = \frac{1}{x}\, dx$$

P67 $y'' - \frac{2}{x} y' = 18x^4$

Set $v = y'$. Then the equation becomes

(1) $v' - \frac{2}{x} v = 18x^4 \rightarrow$ 1st order linear

Integrating factor: I s.t. $I' = -\frac{2}{x} I$

We take $I(x) = \frac{1}{x^2}$. Then

$$(1) \iff \left(\frac{1}{x^2} v\right)' = 18x^2$$

$$\iff \frac{1}{x^2} v = 6x^3 + c$$

$$\iff v = 6x^5 + c_1 x^2$$

Then

$$y = \int v\,dx = x^6 + c_2 x^3$$

P68 System in $\mathbb{R}^3$. We have

$$A^{\#} = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^4 & k^2 \end{bmatrix}$$

$$\overset{A_{12}(-2),\ A_{13}(-1)}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -k^4+1 & k^2-3 \end{bmatrix}$$

$$\overset{A_{23}(1)}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -k^4+4 & k^2-2 \end{bmatrix}$$

(i) If $k^4 \neq 4$ i.e $k \neq \pm\sqrt{2}$ then $\text{Rank}(A) = 3$ and there exists a unique solution

(ii) If $k = \pm\sqrt{2}$ then $k^2 = 2$. Therefore the last row is all 0's and we get an ∞ # of solutions.

P69 Applying the usual formula for 3×3 determinants we get

$\det(A) = 112$ $\det(B) = 36$

Thus

$$\det(B^2A^{-1}) = \frac{(\det(B))^2}{\det(A)} = \frac{36^2}{112} = \frac{81}{7}$$

Pb10 We have $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

$$A^{-1}(3,1) = \frac{1}{\det(A)} [\mathrm{adj}(A)](3,1)$$

$$= \frac{1}{\det(A)} M_c(1,3) = \frac{1}{\det(A)} M_{13}$$

We have

$$\det(A) = 7 \qquad M_{13} = -5$$

Thus

$$A^{-1}(3,1) = -\frac{5}{7}$$

Pb 11

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 5 & 11 & 21 \\ 3 & 7 & 13 \end{bmatrix} \underset{\sim}{\overset{A_{12}(-5)}{A_{13}(-3)}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\overset{A_{23}(-1)}{\sim} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We can deduce

. A non invertible

. $\dim(\text{colspace}(A)) = \text{Rank}(A) = 2.$

. Basis of colspace(A):

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 11 \\ 7 \end{pmatrix} \right\}$$

P6 12 In order to reduce the size of the problem we can

(i) remove $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ (ii) Note that $\begin{pmatrix} 2 \\ 0 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

We are left with colspace(A) with

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \\ 2 & -1 & -1 \end{bmatrix} \underset{\sim}{\overset{A_{12}(-1)\ A_{13}(1)\ A_{14}(-2)}{}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{bmatrix}$$

$$\underset{\sim}{M_2(-1)} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 3 \\ 0 & -3 & -5 \end{bmatrix} \underset{\sim}{\overset{A_{23}(-2)\ A_{24}(3)}{}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{\overset{M_3(-1)\ A_{34}(-1)}{}} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We get 3 leading ones and thus

$\dim(\operatorname{span}\{v_1, .., v_5\}) = 3$

Pb13 $\quad p_1(x) = 1 - ax \qquad p_2(x) = 1 + x$

In P_1, p_1 and p_2 are two vectors with coordinates

$$\begin{pmatrix} 1 \\ -a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

They are linearly independent iff

$$\begin{vmatrix} 1 & 1 \\ -a & 1 \end{vmatrix} \neq 0$$

$\iff 1 + a \neq 0$

$\iff a \neq -1$

Pb 14

For systems: the solution set of a system of equations is a vector space iff the system is linear and homogeneous

Ex in M_2: matrices A such that

$$2a_{11} - 3a_{12} + a_{21} + a_{22} = 0$$

Counter-ex in M_2: (i) Matrices A such that

$$2A - \begin{bmatrix} 1 & 7 \\ 2 & 5 \end{bmatrix} = 0$$

(ii) Matrices such that

$$\det(3A) = 5$$

Ex for differential equations: solutions of

$$y'' - 7y' + \frac{1}{x} y = 0$$

Counter-ex for differential equations: solutions of

$$y'' - 7y' + \frac{1}{x} y = x^3$$

Pb 15 We are given: $T(v_1+v_2) = 3v_1 - v_2$

$T(2v_1+v_2) = v_1+2v_2$

We have

$$v_1 = (2v_1+v_2) - (v_1+v_2)$$

Thus

$$\begin{aligned} T(v_1) &= T(2v_1+v_2) - T(v_1+v_2) \\ &= 3v_1 - v_2 - (v_1+2v_2) \\ &= 2v_1 - 3v_2 \end{aligned}$$

In the same way

$$v_2 = (v_1+v_2) - v_1$$

$$\Rightarrow T(v_2) = T(v_1+v_2) - T(v_1)$$

$$= 3v_1 - v_2 - (2v_1 - 3v_2) = v_1 + 2v_2$$

Then

$$T(av_1+bv_2) = (2a+b)v_1 + (-3a+2b)v_2$$

Pb 16 $\quad A = \begin{pmatrix} 6 & 3 & -4 \\ -5 & -2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$. Expanding w.r.t. the 3rd row we have

$$\det(A-\lambda I) = (-1-\lambda)\left((\lambda-6)(\lambda+2)+15\right)$$
$$= -(\lambda+1)(\lambda^2-4\lambda+3)$$
$$= -(\lambda+1)(\lambda-1)(\lambda-3)$$

We have 3 simple eigenvalues

$\Rightarrow$ A is nondefective.

P6 17

$$A = \begin{bmatrix} 1 & -2 & 5 \\ 2 & -3 & -1 \\ 5 & -8 & -1 \end{bmatrix}$$

$$\overset{A_{12}(-2)}{\underset{A_{13}(-5)}{\sim}} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -11 \\ 0 & 2 & -26 \end{bmatrix}$$

$$\overset{A_{23}(-2)}{\sim} \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -11 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\overset{M_3(-\frac{1}{4})}{\sim} \begin{bmatrix} \boxed{1} & -2 & 5 \\ 0 & \boxed{1} & -11 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

We get 3 leading 1's.
Thus

$$\dim(\ker(T)) = 0 \qquad \dim(\text{Rng}(T)) = 3$$

$$\dim(\ker(T)) + 2\dim(\text{Rng}(T)) = 6$$

Pb 18

$$y'' + 4y = 0 \qquad y(0)=1 \quad y'(0)=1$$

General solution:

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

$$\Rightarrow \quad y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$$

With initial data:

$$\begin{cases} c_1 = 0 \\ 2c_2 = 1 \end{cases}$$

Thus the unique solution is

$$y = \frac{1}{2} \sin(2x)$$

$y_1 = \frac{1}{x}$

Pb 19 $\quad 2x^2y'' + 3xy' - y = 0$

$y_1' = -\frac{1}{x^2}$

$y_1'' = \frac{2}{x^3}$

Let $y_2 = v y_1$. Then (reduction of order method)

$$\left.\begin{aligned} \times(-1) \quad & y_2 = v y_1 \\ \times(3x) \quad & y_2' = v y_1' + v' y_1 \\ \times(2x^2) \quad & y_2'' = v y_1'' + 2v' y_1' + v'' y_1 \end{aligned}\right\} \Rightarrow \begin{aligned} & 2x^2y_2'' + 3xy_2' - y_2 \\ & = v'(3xy_1 + 4x^2y_1') \\ & \quad + v''(2x^2y_1) \end{aligned}$$

We set $\omega = v'$. We obtain, if y_2 solves the eq,

$$2x\omega' - \omega = 0$$

$$\Leftrightarrow \omega' - \frac{1}{2x}\omega = 0 \quad \rightarrow \text{separable eq}$$

We get $\omega = c_1\sqrt{x}$. Therefore

$$v = \int \omega \, dx = c_2 x^{3/2} + c_3$$

and

$$y_2 = v y_1 = c_2 x^{\frac{1}{2}} + \frac{c_3}{x}$$

We take as a fundamental solution

$$y_2 = \sqrt{x}$$

Pb 20 $y'' + y = \tan(t)$

For the homogeneous eq, the fundamental solutions are $y_1 = \cos(t)$ $y_2 = \sin(t)$
$\Rightarrow$ $y_1' = -\sin(t)$ $y_2' = \cos(t)$

Variation of parameter: we look for $y_p = u_1 y_1 + u_2 y_2$, where u_1, u_2 satisfy

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = \tan(t) \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 \\ \tan(t) \end{pmatrix}$$

Thus

$$u_1' = -\frac{\sin^2(t)}{\cos(t)} = -\sec(t) + \cos(t)$$

$$u_2' = \sin(t)$$

We get

$$u_1 = \ln(|\sec(t) + \tan(t)|), \quad u_2 = -\cos(t)$$

and

$$y_p = [\ln(|\sec(t) + \tan(t)|) - \sin(t)]\cos(t)$$

P6.21 $y^{(4)} - y = 3t + \cos(t)$

Auxiliary pol: $P(r) = r^4 - 1 = (r^2-1)(r^2+1)$
$= (r+1)(r-1)(r+i)(r-i)$

Therefore y_p will be of the form

$$y_p = (a_1 t + a_2) + (a_3 t + a_4)\cos(t) + (a_5 t + a_6)\sin(t)$$

Pb 22 $\quad x' = \underbrace{\begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}}_{A} x \qquad x(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

Eigenvalue decomposition for A:

$\lambda_1 = 2 \qquad v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\lambda_2 = 4 \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus general solution:

$$x = c_1 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

With initial condition:

$$\begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = \frac{-3}{2} \\ c_2 = \frac{7}{2} \end{matrix}$$

Unique solution:

$$x = -\frac{3}{2} e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{7}{2} e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

P6 23 $\quad x' = \overbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}^{A} x + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$

Eigenvalue decomposition of A:

$\lambda_1 = -3 \quad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}; \quad \lambda_2 = -1 \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Thus $X(t) = \begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix}$.

We have $x_p(t) = X(t)u(t)$, where u' solves $Xu' = b$. We get

$$\begin{pmatrix} -e^{-3t} & e^{-t} \\ e^{-3t} & e^{-t} \end{pmatrix} u' = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}$$

$$\Leftrightarrow \quad u' = \begin{pmatrix} -e^{2t} + \frac{3}{2}t e^{3t} \\ \frac{3}{2}te^{t} + 1 \end{pmatrix}$$

Therefore

$$u = \begin{pmatrix} -\frac{1}{2}e^{2t} + \frac{1}{6}(3t-1)e^{3t} \\ \frac{3}{2}(t-1)e^{t} + t \end{pmatrix}$$

and

$$x_p = X(t)u(t)$$