Systems of linear differential equations

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Differential equations and linear algebra - MA 262

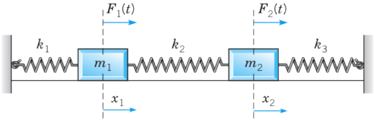
Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

Outline

Spring example

Physical setting: Interacting springs



Equation:

$$
\begin{aligned}
m_1 \frac{d^2 x_1}{dt^2} &= k_2(x_2 - x_1) - k_1 x_1 + F_1(t) \\
m_2 \frac{d^2 x_2}{dt^2} &= -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)
\end{aligned}
$$

Second order equation as first order system (1)

Equation:

$$y'' + 0.125y' + y = 0$$

Aim:

Write this equation as a system of differential equations

Second order equation as first order system (2)

Equation:

$$y'' + 0.125y' + y = 0$$

Change of variable: set

$$x_1 = y, \qquad x_2 = y'$$

New equation:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 0.125x_2 \end{aligned}$$

First order system as second order equation (1)

System:

$$\begin{aligned} x' &= -2y \\ y' &= \frac{1}{2}x \end{aligned}$$

Aim:

Write this system as a second order differential equation

First order system as second order equation (2)

Differentiating x: We get

$$x'' = -2y' = -x, \quad \text{thus} \quad x'' + x = 0$$

General solution for x:

$$x(t) = A\cos(t) + B\sin(t) = C\cos(t - \varphi)$$

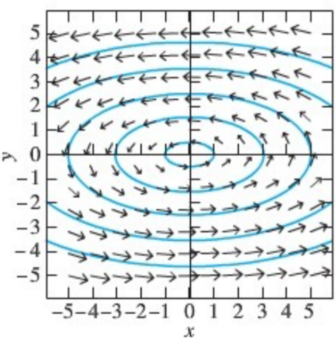
General solution for y:

$$y(t) = -\frac{1}{2}x'(t) = \frac{C}{2}\sin(t - \varphi)$$

First order system as second order equation (3)

General solution

$$\begin{aligned} x(t) &= C\cos(t-\varphi) \\ y(t) &= \frac{C}{2}\sin(t-\varphi) \end{aligned}$$



Another example of first order system (1)

System:

$$
\begin{aligned}
x' &= y \\
y' &= 2x + y
\end{aligned}
$$

Aim:

Write this system as a second order differential equation

Another example of first order system (2)

Differentiating x: We get

$$x'' = y' = 2x + y = x' + 2x, \quad \text{thus} \quad x'' - x' - 2x = 0$$

General solution for x:

$$x(t) = A\,e^{-t} + B\,e^{2t}$$

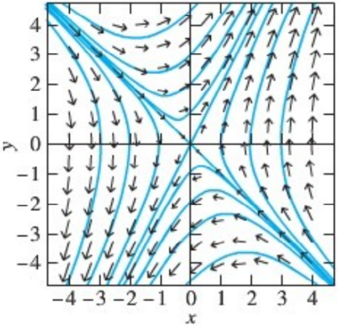
General solution for y:

$$y(t) = x'(t) = -A\,e^{-t} + 2B\,e^{2t}$$

Another example of first order system (3)

General solution

$$\begin{aligned} x(t) &= A\,e^{-t} + B\,e^{2t} \\ y(t) &= -A\,e^{-t} + 2B\,e^{2t} \end{aligned}$$



Definitions

First order linear system: Of the form

$$\begin{array}{lllllll} x_1'(t) & = a_{11}(t)x_1(t) & +a_{12}(t)x_2(t) & + & \cdots & +a_{1n}(t)x_n(t) & +f_1(t) \\ x_2'(t) & = a_{21}(t)x_1(t) & +a_{22}(t)x_2(t) & + & \cdots & +a_{2n}(t)x_n(t) & +f_2(t) \\ \vdots & & & & & & \vdots \\ x_n'(t) & = a_{n1}(t)x_1(t) & +a_{n2}(t)x_2(t) & + & \cdots & +a_{nn}(t)x_n(t) & +f_n(t) \end{array}$$

Homogeneous system: When

$$f_1 = f_2 = \cdots = f_n = 0$$

Nonhomogeneous system: When there exists j such that

$$f_j \neq 0$$

Initial value

Definition 1.

For the system above an initial condition is given by

$$x_1(t_0) = x_{1,0}, \ldots, x_n(t_0) = x_{n,0}$$

Example of system:

$$\begin{aligned} x_1' &= x_1 + 2x_2 \\ x_2' &= 2x_1 - 2x_2 \end{aligned}$$

Initial condition:

$$x_1(0) = 1, \qquad x_2(0) = 0$$

Example of initial value

Form of the general solution: We will see that

$$x_1(t) = c_1\, e^{-3t} + c_2\, e^{2t}, \quad \text{and} \quad x_2(t) = -2c_1\, e^{-3t} + \frac{1}{2} c_2\, e^{2t}$$

System for c_1, c_2:

$$\begin{array}{rrl} c_1 & +c_2 & = 1 \\ -4c_1 & +c_2 & = 0 \end{array}$$

Unique solution of the initial value problem:

$$x_1(t) = \frac{1}{5}\, e^{-3t} + \frac{4}{5}\, e^{2t}, \quad \text{and} \quad x_2(t) = -\frac{2}{5}\, e^{-3t} + \frac{2}{5}\, e^{2t}$$

Outline

Matrix notation

First order linear system: Of the form

$$\begin{array}{rcccccl} x_1'(t) & = a_{11}(t)x_1(t) & +a_{12}(t)x_2(t) & + & \cdots & +a_{1n}(t)x_n(t) & +f_1(t) \\ x_2'(t) & = a_{21}(t)x_1(t) & +a_{22}(t)x_2(t) & + & \cdots & +a_{2n}(t)x_n(t) & +f_2(t) \\ \vdots & & & & & & \vdots \\ x_n'(t) & = a_{n1}(t)x_1(t) & +a_{n2}(t)x_2(t) & + & \cdots & +a_{nn}(t)x_n(t) & +f_n(t) \end{array}$$

Related matrices:

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & a_{n3}(t) & \dots & a_{nn}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

Matrix notation (2)

Vector of unknown: We set

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}$$

Vector form of the linear system:

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t) + \mathbf{f}(t)$$

Initial data:

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

Some vector space notions

Space $V_n(I)$: For an interval I we set

$$V_n(I) = \{y : I \to \mathbb{R}^n\}.$$

Then $V_n(I)$ is a vector space.

Wronskian: Let

- $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ vectors in $V_n(I)$

The Wronskian of those vectors is

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det\left([\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)]\right)$$

Wronskian and independence

Theorem 2.

Let

- $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ vectors in $V_n(I)$.
- Assume that $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$ for a given $t_0 \in I$

Then

$$\{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \text{ is linearly independent.}$$

Example of Wronskian

Vector function:

$$\mathbf{x}_1(t) = \begin{bmatrix} e^t \\ 2e^t \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} 3\sin(t) \\ \cos(t) \end{bmatrix}$$

Wronskian: We have

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^t & 3\sin(t) \\ 2e^t & \cos(t) \end{vmatrix} = e^t\,(\cos(t) - 6\sin(t))$$

Linear independence: We have

$$W[\mathbf{x}_1, \mathbf{x}_2](0) = 1 \neq 0.$$

Therefore $\{\mathbf{x}_1, \mathbf{x}_2\}$ is linearly independent

JÃşzef Maria Hoene-Wroński

Wronski: A philosopher-mathematician

- Born in Poland (1776)
- Lived mostly in France
- Hero of the Polish army
  when defeated by the Russians
- Mathematician
  Wronskian is his main contribution
- Philosophical system based on math
- Ousted from the observatory
  because of his philosophical views
- Died in poverty, aged 76

Homogeneous equation

Theorem 3.

Consider the system

$$\mathbf{x}'(t) = A(t)\,\mathbf{x}(t), \qquad \mathbf{x}(t) \in \mathbb{R}^n,\ A(t) \in M_{n,n}.$$

Hypothesis:

The mapping $t \mapsto A(t)$ is continuous

Then the following holds true:

1. The general solution set is a vector space of dimension n
2. The system with initial data $\mathbf{x}(t_0) = \mathbf{x}_0$ admits a unique solution

Fundamental solutions

Definition 4.

Consider

- The system $\mathbf{x}'(t) = A(t)\,\mathbf{x}(t)$
- A set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of n linearly independent solutions of the system

Then:

1. The set
$$\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$$
is called fundamental solution set of the system
2. The matrix
$$\mathbf{X}(t) = [\mathbf{x}_1, \ldots, \mathbf{x}_n]$$
is called fundamental matrix of the system

Wronskian and fundamental solutions

Theorem 5.

Consider

- The system $\mathbf{x}'(t) = A(t)\,\mathbf{x}(t)$ on an interval I
- A set $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ of n solutions of the system
- $t_0 \in I$

Then:

1. If $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$ then $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a fundamental solution set of the system
2. The general solution of the system can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$$

Example of application

System under consideration:

$$\mathbf{x}' = A\mathbf{x}, \quad \text{with} \quad A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \tag{1}$$

Solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} -e^t \cos(2t) \\ e^t \sin(2t) \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} e^t \sin(2t) \\ e^t \cos(2t) \end{bmatrix}$$

Remark:

One can check that $\mathbf{x}_1$ and $\mathbf{x}_2$ solve (1)

Example of application (2)

Wronskian computation:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} -e^t \cos(2t) & e^t \sin(2t) \\ e^t \sin(2t) & e^t \cos(2t) \end{vmatrix} = -e^{2t}$$

Conclusion: Since $W[\mathbf{x}_1, \mathbf{x}_2](t) \neq 0$ for all $t \in \mathbb{R}$,

$$\{\mathbf{x}_1, \mathbf{x}_2\} \text{ is a fundamental solution set}$$

General form of the solution to (1):

$$\mathbf{x}(t) = \begin{bmatrix} e^t \left(-c_1 \cos(2t) + c_2 \sin(2t)\right) \\ e^t \left(c_1 \sin(2t) + c_2 \cos(2t)\right) \end{bmatrix}$$

Outline

Aim

General objective: Solve homogeneous systems of the form

$$\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t),$$

with

$$\mathbf{x}(t) \in \mathbb{R}^n, \qquad \mathbf{A} \in M_{n,n}.$$

Methodology:
Based on eigenvalues/eigenvectors decomposition of $\mathbf{A}$

Solutions and eigenvectors

Theorem 6.

Consider the system with constant matrix

$$\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t), \qquad \mathbf{x}(t) \in \mathbb{R}^n, \ \mathbf{A} \in M_{n,n}. \tag{2}$$

Hypothesis:

- $\mathbf{A}$ admits n lin. independ. eigen. $\mathbf{u}_k$ with eigenval. λ_k

Conclusion:

1. The following are linearly independent solutions to (2):

$$\mathbf{x}_k(t) = e^{\lambda_k t}\mathbf{u}_k$$

2. The general solution of (2) is of the form

$$\mathbf{x}(t) = c_1\, e^{\lambda_1 t}\mathbf{u}_1 + \cdots + c_n\, e^{\lambda_n t}\mathbf{u}_n$$

Outline

Example with real eigenvalues

Equation:

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_1 = 3, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \lambda_2 = -1, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example with real eigenvalues (2)

Fundamental solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}, \qquad \mathbf{x}_2(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

Wronskian:

$$W[\mathbf{x}_1, \mathbf{x}_2](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Conclusion: $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent

Example with real eigenvalues (3)

General solution:

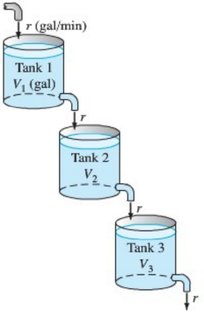
$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-t}.$$

Compartmental analysis (1)

Situation:

- Three brine tanks, volume V_1, V_2, V_3
- Fresh water flows into tank 1, rate r
- Mixed water flows from tank 2 into tank 3, rate r
- Mixed water flows out of tank 3, rate r

Aim: Compute quantity of salt in each tank i



Compartmental analysis (2)

Notation: Set

$$k_i = \frac{r}{V_i}$$

Equations:

$$\begin{aligned} x_1' &= -k_1 x_1 \\ x_2' &= k_1 x_1 - k_2 x_2 \\ x_3' &= k_2 x_2 - k_3 x_3 \end{aligned}$$

Compartmental analysis (3)

Specific values for the volumes: Take

$$V_1 = 20, \qquad V_2 = 40, \qquad V_3 = 50$$

Specific values for the rate: Take

$$r = 10$$

Initial value: Assume

$$x_1(0) = 15, \qquad x_2(0) = 0, \qquad x_3(0) = 0$$

Compartmental analysis (4)

System under consideration:

$$\mathbf{x}' = A\mathbf{x}, \quad \text{with} \quad A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0.5 & -0.25 & 0 \\ 0 & 0.25 & -0.2 \end{bmatrix} \tag{3}$$

General solution:

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 3 \\ -6 \\ 5 \end{bmatrix} e^{-t/2} + c_2 \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix} e^{-t/4} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t/5}.$$

Compartmental analysis (5)

Initial value: With $x_1(0) = 15$, $x_2(0) = 0$, $x_3(0) = 0$, we get

$$\begin{array}{rrrcr} 3c_1 & & & = & 15 \\ -6c_1 & +c_2 & & = & 0 \\ 5c_1 & -5c_2 & +c_3 & = & 0 \end{array}$$

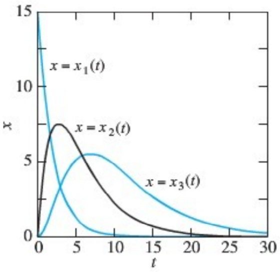
Values for the constants:

$$c_1 = 5, \qquad c_2 = 30, \qquad c_3 = 125$$

Compartmental analysis (6)

Particular solution:

$$\mathbf{x}(t) = \begin{bmatrix} 15 \\ -30 \\ 25 \end{bmatrix} e^{-t/2} + \begin{bmatrix} 0 \\ -30 \\ 30 \end{bmatrix} e^{-t/4} + \begin{bmatrix} 0 \\ 0 \\ 125 \end{bmatrix} e^{-t/5}.$$



Outline

Method for complex eigenvalues

Theorem 7.

Consider the system with constant matrix

$$\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t), \qquad \mathbf{x}(t) \in \mathbb{R}^n,\ \mathbf{A} \in M_{n,n}. \tag{4}$$

Hypothesis: We have complex eigenvalues/eigenvectors

$$\lambda = \alpha \pm \imath\beta \quad \text{and} \quad \mathbf{u} = \mathbf{a} \pm \imath\mathbf{b}.$$

Conclusion: We have 2 real valued independent solutions to (4)

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\alpha t}\left(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}\right) \\ \mathbf{x}_2(t) &= e^{\alpha t}\left(\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}\right). \end{aligned}$$

Example with complex eigenvalues

Equation:

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_1 = -\frac{1}{2} + \imath, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ \imath \end{bmatrix}; \qquad \lambda_2 = -\frac{1}{2} - \imath, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ -\imath \end{bmatrix}$$

Example with complex eigenvalues (2)

Fundamental solutions:

$$\mathbf{x}_1(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} e^{-\frac{1}{2}t}$$

$$\mathbf{x}_2(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} e^{-\frac{1}{2}t}.$$

Remark: Only $\lambda_1, \mathbf{u}_1$ are used in order to compute $\mathbf{x}_1$ and $\mathbf{x}_2$

Outline

Example of matrix with repeated root

Matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Characteristic polynomial:

$$P_{\mathbf{A}}(r) = \det(\mathbf{A} - r\,\mathrm{Id}) = (r-2)^2$$

Eigenvalues and eigenvectors:

$$r = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Remark: $r = 2$ is a double eigenvalue, with 1 eigenvector only.

Generalized eigenvectors with multiplicity 2

System: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} \in \mathbb{R}^{2,2}$ and $\det(\mathbf{A}) \neq 0$

Situation:

- $\mathbf{A}$ has a double eigenvalue r
- Unique eigenvector $\mathbf{v}$ (up to constant factor)

Recipe to find generalized eigenvectors:

1. Find $\mathbf{v}_2$ such that $(\mathbf{A} - r\,\mathrm{Id})^2\mathbf{v}_2 = \mathbf{0}$, but not parallel to $\mathbf{v}$
2. Compute $\mathbf{v}_1 = (\mathbf{A} - r\,\mathrm{Id})\mathbf{v}_2$
3. Then $\mathbf{v}_1, \mathbf{v}_2$ are generalized eigenvectors

Solving systems with multiplicity 2

Situation:

- We consider the system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$
- $\mathbf{A}$ has a double eigenvalue r
- Generalized eigenvectors $\mathbf{v}_1, \mathbf{v}_2$

Corresponding fundamental solutions: We get

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{v}_1 e^{rt} \\ \mathbf{x}_2(t) &= (\mathbf{v}_1 t + \mathbf{v}_2)\, e^{rt} \end{aligned}$$

Example with multiplicity 2 (1)

Equation:

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

Eigenvalues and eigenvector:

$$r = 2 \;\; \text{(multiplicity 2)}, \qquad \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Square of a matrix: We have

$$\mathbf{A} - 2\,\mathrm{Id} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad (\mathbf{A} - 2\mathrm{Id})^2 = \mathbf{0}$$

Example with multiplicity 2 (2)

Applying the recipe to find the generalized eigenvectors: We choose

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_1 = (\mathbf{A} - 2\mathrm{Id})\,\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

General solution:

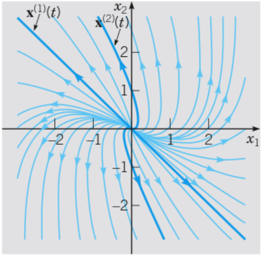
$$\mathbf{x} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} te^{2t} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$$

Example with multiplicity 2 (3)

Asymptotic behavior: As $t \to \infty$

- $\mathbf{x}(t) \to \infty$
- $\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = -1$, thus slope $\simeq -1$
- $\mathbf{x}(t)$ does not approach the asymptote

Graph in the x_1x_2 plane:



Generalized eigenvectors with multiplicity 3

Situation:

- $\mathbf{A}$ has a triple eigenvalue r
- Unique eigenvector $\mathbf{v}$ (up to constant factor)

Recipe to find generalized eigenvectors:

1. Find $\mathbf{v}_3$ such that $(\mathbf{A} - r\,\text{Id})^3\mathbf{v}_3 = \mathbf{0}$, not parallel to $\mathbf{v}$
2. Compute $\mathbf{v}_2 = (\mathbf{A} - r\,\text{Id})\mathbf{v}_3$
3. Compute $\mathbf{v}_1 = (\mathbf{A} - r\,\text{Id})\mathbf{v}_2$
4. Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are generalized eigenvectors

Solving systems with multiplicity 3

Situation:

- We consider the system $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$
- $\mathbf{A}$ has a triple eigenvalue r
- Generalized eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

Corresponding fundamental solutions: We get

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{v}_1 e^{rt} \\ \mathbf{x}_2(t) &= (\mathbf{v}_1 t + \mathbf{v}_2)\, e^{rt} \\ \mathbf{x}_3(t) &= \left(\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3\right) e^{rt} \end{aligned}$$

Example with multiplicity 3 (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Aim:

Expression of the general solution to this system

Example with multiplicity 3 (2)

Eigenvalues and eigenvector:

$$r = -1 \;\; \text{(multiplicity 3)}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Example with multiplicity 3 (3)

Third power computation: We find

$$(A + \mathrm{Id})^3 = \mathbf{0}$$

Value for $\mathbf{v}_3$: We take

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example with multiplicity 3 (4)

Value for $\mathbf{v}_2$: We compute

$$\mathbf{v}_2 = (A + \mathrm{Id})\mathbf{v}_3 = \begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We get

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

Example with multiplicity 3 (5)

Checking value for $\mathbf{v}_1$: We compute

$$\mathbf{v}_1 = (A + \mathrm{Id})\mathbf{v}_2 = \begin{bmatrix} 1 & 1 & 2 \\ -5 & -2 & -7 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}$$

We get

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} = -2\,\mathbf{v}$$

Example with multiplicity 3 (6)

Fundamental solutions: Recall that

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{v}_1 e^{-t} \\ \mathbf{x}_2(t) &= (\mathbf{v}_1 t + \mathbf{v}_2)\, e^{-t} \\ \mathbf{x}_3(t) &= \left(\frac{1}{2}\mathbf{v}_1 t^2 + \mathbf{v}_2 t + \mathbf{v}_3\right) e^{-t} \end{aligned}$$

Summarizing values of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$: We have found

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Example with multiplicity 3 (7)

Fundamental solutions in our case: We find

$$
\begin{aligned}
\mathbf{x}_1(t) &= \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t} \\
\mathbf{x}_2(t) &= \begin{bmatrix} -2t+1 \\ -2t-5 \\ 2t+1 \end{bmatrix} e^{-t} \\
\mathbf{x}_3(t) &= \begin{bmatrix} -t^2+t+1 \\ -t^2-5t \\ t^2+t \end{bmatrix} e^{-t}
\end{aligned}
$$

Outline

Aim

Brief summary of what we have seen:

- System $\mathbf{x}' = \mathbf{A}\mathbf{x}$
- λ eigenvalue with eigenvector $\mathbf{v}$

Then a solution to the system is

$$\mathbf{x}(t) = \mathbf{v}\, e^{\lambda t}$$

Next step:
Geometric interpretations of the eigenvalue decomposition

Summary in a 2-d situation

Theorem 8.

System: $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$, with $\mathbf{A} \in M_2$

Then we have 3 cases:

1. 2 distinct real eigenvalues: General solution of the form

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t}$$

2. 2 distinct complex eigenvalues: General solution

$$\mathbf{x} = c_1 e^{\alpha t} \left(\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}\right) + c_2 e^{\alpha t} \left(\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}\right)$$

3. Repeated eigenvalue: General solution of the form

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda t} + c_2 \left(\mathbf{v}_1 t + \mathbf{v}_2\right) e^{\lambda t}$$

Outline

Signs for the eigenvalues

Real distinct eigenvalues: We will distinguish 5 cases

- Nonzero of opposite sign: $\lambda_1 < 0 < \lambda_2$
- Both negative: $\lambda_1 < \lambda_2 < 0$
- Both positive: $0 < \lambda_2 < \lambda_1$
- One zero, one negative: $\lambda_1 < \lambda_2 = 0$
- One zero, one positive: $0 = \lambda_2 < \lambda_1$

Repeated eigenvalue: We will distinguish 3 cases

- Positive: $\lambda_1 = \lambda_2 > 0$
- Negative: $\lambda_1 = \lambda_2 < 0$
- Zero: $\lambda_1 = \lambda_2 = 0$

Saddle points: $\lambda_1 < 0 < \lambda_2$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_1 = -2, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}; \qquad \lambda_2 = 5, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General solution:

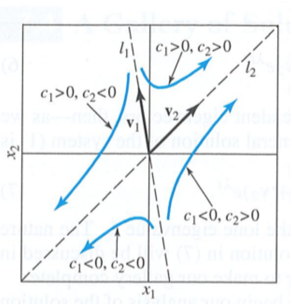
$$\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 6 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

Saddle points: $\lambda_1 < 0 < \lambda_2$ (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 6 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$

Geometric information:

- As $t \to \infty$, $\mathbf{v}_2$ is the asymptotic direction
- Quadrant in which $\mathbf{x}$ is located: according to c_1, c_2



Sinks: $\lambda_1 < \lambda_2 < 0$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_1 = -14, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \qquad \lambda_2 = -7, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

General solution:

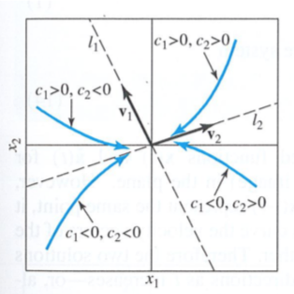
$$\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-14t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-7t}$$

Sinks: $\lambda_1 < \lambda_2 < 0$ (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-14t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-7t}$

Geometric information:

- As $t \to \infty$, $\mathbf{x}(t) \to \mathbf{0}$
- If $c_2 \neq 0$, as $t \to \infty$ $\mathbf{x}'$ is closer to the direction of $\mathbf{v}_2$
- Quadrant in which $\mathbf{x}$ is located: according to c_1, c_2



Sources: $0 < \lambda_2 < \lambda_1$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 8 & -3 \\ -2 & 13 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_2 = 7, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}; \qquad \lambda_1 = 14, \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

General solution:

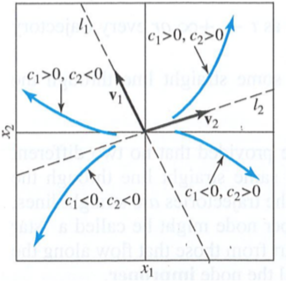
$$\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{14t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{7t}$$

Sources: $0 < \lambda_2 < \lambda_1$ (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{14t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{7t}$

Geometric information:

- As $t \to \infty$, $\mathbf{x}(t) \to \infty$
- If $c_2 \neq 0$, as $t \to -\infty$ $\mathbf{x}'$ is closer to the direction of $\mathbf{v}_2$
- Quadrant in which $\mathbf{x}$ is located: according to c_1, c_2



Line solutions: $\lambda_1 < \lambda_2 = 0$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} -36 & -6 \\ 6 & 1 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_1 = -35, \quad \mathbf{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}; \qquad \lambda_2 = 0, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

General solution:

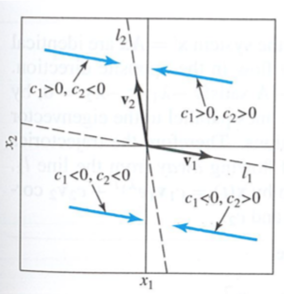
$$\mathbf{x} = c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{-35t} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

Line solutions: $\lambda_1 < \lambda_2 = 0$ (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{-35t} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

Geometric information:

- As $t \to \infty$, $\mathbf{x}(t) \to c_2 \mathbf{v}_2$
- The solution converges to a constant vector as $t \to \infty$
- Quadrant in which $\mathbf{x}$ is located: according to c_1, c_2



Line solutions: $0 = \lambda_2 < \lambda_1$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 36 & 6 \\ -6 & -1 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$\lambda_1 = 35, \quad \mathbf{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}; \qquad \lambda_2 = 0, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

General solution:

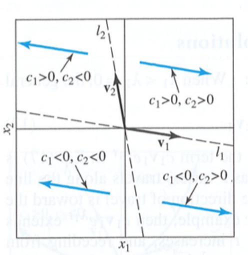
$$\mathbf{x} = c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{35t} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

Line solutions: $0 = \lambda_2 < \lambda_1$ (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} e^{35t} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

Geometric information:

- As $t \to \infty$, $\mathbf{x}(t) \to c_2 \mathbf{v}_2$
- The solution converges to a constant vector as $t \to -\infty$
- As $t \to \infty$, solutions are flowing away from $\mathbf{v}_2$
  $\hookrightarrow$ in the direction of $\mathbf{v}_1$
- Quadrant in which $\mathbf{x}$ is located: according to c_1, c_2



Repeated eigenvalue with 2 eigenvectors (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition: Double eigenvalue,

$$\lambda = 2, \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

General solution:

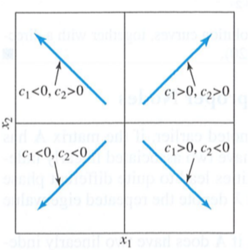
$$\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{2t}$$

Repeated eigenvalue with 2 eigenvectors (2)

General solution: $\mathbf{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{2t}$

Geometric information:

- Solutions are rays
- As $t \to \infty$, solutions are flowing away from 0
  $\hookrightarrow$ in the direction of (c_1, c_2)
- Quadrant in which $\mathbf{x}$ is located: according to c_1, c_2



Repeated eigenvalue with 1 eigenvector (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition: Double eigenvalue $\lambda = 4$,

$$\text{Eigenvector } \mathbf{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \quad \text{Generalized eigenvector } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

General solution:

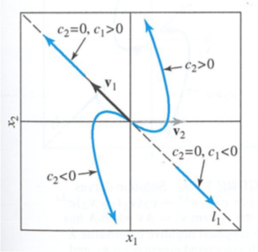
$$\mathbf{x} = c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -3t + 1 \\ 3t \end{bmatrix} e^{4t}$$

Repeated eigenvalue with 1 eigenvector (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} e^{4t}$

Geometric information:

- $\lim_{t\to-\infty} x(t) = 0$, along the direction of $\mathbf{v}_1$
- As $t \to \infty$, solutions are flowing away from 0
  $\hookrightarrow$ along the direction of $\mathbf{v}_1$
- Half plane in which $\mathbf{x}$ is located: according to c_2



Repeated eigenvalue with 1 eigenvector (3)

Another geometric information:

- For all curves, the tangent at 0 is $\mathbf{v}_1$

Terminology:

This case is called improper nodal source

Repeated 0 eigenvalue with 1 eigenvector (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix} \mathbf{x}$$

Eigenvalue decomposition: Double eigenvalue $\lambda = 0$,

$$\text{Eigenvector } \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \text{Generalized eigenvector } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

General solution:

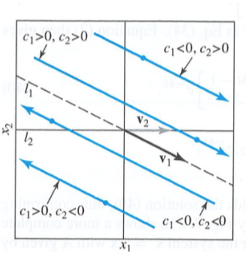
$$\mathbf{x} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2t+1 \\ -t \end{bmatrix}$$

Repeated 0 eigenvalue with 1 eigenvector (2)

General solution: $\mathbf{x} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 2t+1 \\ -t \end{bmatrix}$

Geometric information:

- $\mathbf{x}$ line parallel to $\mathbf{v}_1$
- Starting point for $t = 0$: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$



Outline

3 main situations

Cases to be distinguished for λ_1:

- Pure imaginary:
  $\lambda_1 = \imath q$ with $q \neq 0$
- Complex with negative real part:
  $\lambda_1 = p + \imath q$ with $p < 0$ and $q \neq 0$
- Complex with positive real part:
  $\lambda_1 = p + \imath q$ with $p > 0$ and $q \neq 0$

Note:
We also have $\lambda_2 = \bar{\lambda}_1$

Elliptic solutions: $\lambda_1 = \imath q$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Eigenvalue decomposition:

$$\lambda_1 = 10\imath, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \imath \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

General solution:

$$\begin{aligned} \mathbf{x} &= c_1 \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cos(10t) - \begin{bmatrix} 5 \\ 0 \end{bmatrix} \sin(10t) \right) \\ &\quad + c_2 \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix} \cos(10t) + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \sin(10t) \right) \end{aligned}$$

Elliptic solutions: $\lambda_1 = \imath q$ (2)

Initial value: We take

$$\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Computing c_1, c_2: We get

$$c_1 = c_2 = \frac{1}{2}$$

Unique solution:

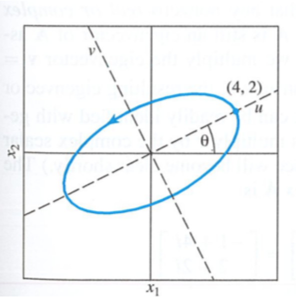
$$\mathbf{x}(t) = \begin{bmatrix} 4\cos(10t) - \sin(10t) \\ 2\cos(10t) + 2\sin(10t) \end{bmatrix}$$

Elliptic solutions: $\lambda_1 = \imath q$ (3)

Unique solution: $\mathbf{x} = \begin{bmatrix} 4\cos(10t) - \sin(10t) \\ 2\cos(10t) + 2\sin(10t) \end{bmatrix}$

Geometric information:

- Solution located on an ellipse
- Goes counterclockwise like the previous ellipse



Spiral solutions: $\lambda_1 = p + \imath q$, with $p < 0$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} 5 & -17 \\ 8 & -7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Eigenvalue decomposition:

$$\lambda_1 = -1 + 10\imath, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \imath \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

General solution:

$$\begin{aligned} \mathbf{x} \;=\; & c_1 e^{-t} \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cos(10t) - \begin{bmatrix} 5 \\ 0 \end{bmatrix} \sin(10t) \right) \\ & + c_2 e^{-t} \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix} \cos(10t) + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \sin(10t) \right) \end{aligned}$$

Spiral solutions: $\lambda_1 = p + \imath q$, with $p < 0$ (2)

Initial value: We take

$$\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Computing c_1, c_2: We get

$$c_1 = c_2 = \frac{1}{2}$$

Unique solution:

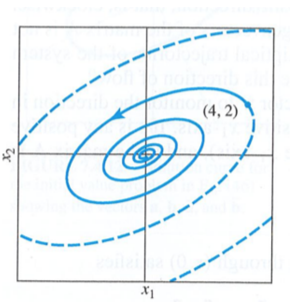
$$\mathbf{x}(t) = \begin{bmatrix} e^{-t}\left(4\cos(10t) - \sin(10t)\right) \\ e^{-t}\left(2\cos(10t) + 2\sin(10t)\right) \end{bmatrix}$$

Spiral solutions: $\lambda_1 = p + \imath q$, with $p < 0$ (3)

Unique solution: $\mathbf{x} = \begin{bmatrix} e^{-t}\,(4\cos(10t) - \sin(10t)) \\ e^{-t}\,(2\cos(10t) + 2\sin(10t)) \end{bmatrix}$

Geometric information:

- Solution located on an "ellipse" reeling in as $t \to \infty$
- Goes counterclockwise: $\mathbf{x}'(0) = 10(-1, 2)^T$



Spiral solutions: $\lambda_1 = p + \imath q$, with $p > 0$ (1)

Equation:

$$\mathbf{x}' = \begin{bmatrix} -5 & 17 \\ -8 & 7 \end{bmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Eigenvalue decomposition:

$$\lambda_1 = 1 + 10\imath, \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \imath \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

General solution:

$$\begin{aligned} \mathbf{x} &= c_1 e^t \left(\begin{bmatrix} 3 \\ 4 \end{bmatrix} \cos(10t) - \begin{bmatrix} 5 \\ 0 \end{bmatrix} \sin(10t) \right) \\ &\quad + c_2 e^t \left(\begin{bmatrix} 5 \\ 0 \end{bmatrix} \cos(10t) + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \sin(10t) \right) \end{aligned}$$

Spiral solutions: $\lambda_1 = p + \imath q$, with $p > 0$ (2)

Initial value: We take

$$\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Computing c_1, c_2: We get

$$c_1 = c_2 = \frac{1}{2}$$

Unique solution:

$$\mathbf{x}(t) = \begin{bmatrix} e^t \left(4\cos(10t) - \sin(10t)\right) \\ e^t \left(2\cos(10t) + 2\sin(10t)\right) \end{bmatrix}$$

Spiral solutions: $\lambda_1 = p + \imath q$, with $p > 0$ (3)

Unique solution: $\mathbf{x} = \begin{bmatrix} e^t\left(4\cos(10t) - \sin(10t)\right) \\ e^t\left(2\cos(10t) + 2\sin(10t)\right) \end{bmatrix}$

Geometric information:

- Solution located on an "ellipse" spiraling away as $t \to \infty$
- Goes clockwise (ellipse has been inverted)