Differential equations: introduction

Samy Tindel

Purdue University

Differential equations - MA 26600

Taken from *Elementary differential equations* by Boyce and DiPrima

Outline

Outline

Interest of differential equations

Differential equation:

Equation in which the derivative of a function appears.

Features of differential equations:

- Theoretical interest
- Always related to a physical system:
  - Fluid dynamics
  - Electrical circuits
  - Population dynamics
  - Economy, finance
- More than 300 years of study
- Still active domain of research

Outline

Expression of Newton's law

Example of physical situation:
Object falling in the atmosphere near sea level

Notation:

- t = time variable, in seconds
- v = velocity, depends on time $v = v(t)$, in $m\,s^{-1}$
- F = force
- a = acceleration

Orientation: Downwards

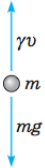
Newton's law:

$$F = m\,a = m\,\frac{dv}{dt}$$

Gravity example (2)

Forces acting on the object:

- Gravity: mg, where $g = 9.81 ms^{-2}$ close to earth
- Air resistance, drag: $-\gamma v$, where γ object dependent



Total force: $F = mg - \gamma v$

Resulting equation:

$$m \frac{dv}{dt} = mg - \gamma v \quad (1)$$

Qualitative study

Specific values for coefficients:

$\hookrightarrow$ We take $m = 10\text{kg}$ and $\gamma = 2\text{kg s}^{-1}$

Specific equation:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{2}$$

Note:

- One can solve equation (2)
- Qualitative study: draw conclusions from equation itself

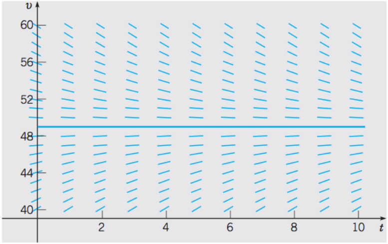
Example of slope:

$\hookrightarrow$ If $v = 40$, then $\frac{dv}{dt} = 1.8$

Direction field

Meaning of the graph:

$\hookrightarrow$ Values of $\frac{dv}{dt}$ according to values of v



What can be seen on the graph:

- Critical value: $v_c = 49\text{ms}^{-1}$, solution to $9.8 - \frac{v}{5} = 0$
- If $v < v_c$: positive slope
- If $v > v_c$: negative slope

Qualitative study (2)

Equilibrium: According to the graph

- $v(t) \equiv v_c$ is solution to (2)
- All solutions converge to v_c as $t \to \infty$

Remark:

1. The facts above will be shown later on
2. v_c is called equilibrium for system (2)

Generalization: For general system (1):

- Equilibrium: $v_c = \frac{mg}{\gamma}$
- Convergence to equilibrium

Outline

Predator-pray model

Situation:

- $p = p(t) =$ mice population
- Reproduction rate for mice: r mice/month
- Presence of owl: k mice eaten per month

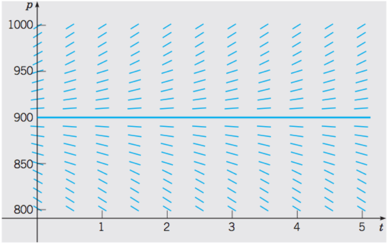
Resulting equation:

$$\frac{dp}{dt} = rp - k$$

Direction field

Meaning of the graph:

$\hookrightarrow$ Values of $\frac{dv}{dt}$ according to values of v



What can be seen on the graph:

- Critical value: $p_c = \frac{k}{r}$, solution to $rp - k = 0$
- If $p < p_c$: negative slope
- If $p > p_c$: positive slope

Qualitative study (2)

Equilibrium: According to the graph

- $p(t) \equiv p_c$ is solution to (2)
- A solution will never converge to p_c as $t \to \infty$
- If $p(0) > p_c$, population increases
- If $p(0) < p_c$, extinction

Remark:

- p_c is an unstable equilibrium for system (2)

Outline

Importance of direction fields

General form of an equation:

$$\frac{dy}{dt} = f(t, y)$$

Conclusions from previous examples:

1. Importance of direction fields graphs $(t, y) \mapsto f(t, y)$
2. Plotting $(t, y) \mapsto f(t, y)$ is easier than solving the equation
3. It can be done with the help of a computer

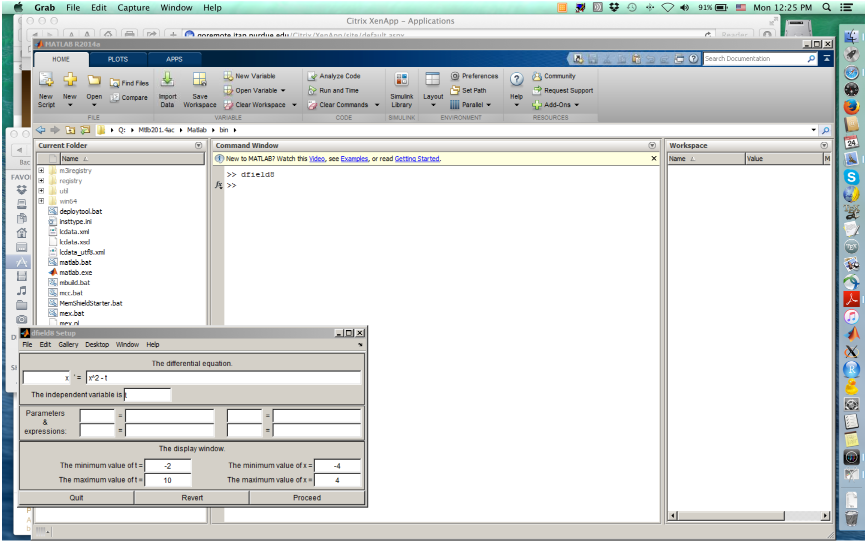
Matlab dfield8 function

Remote connexion to Matlab:

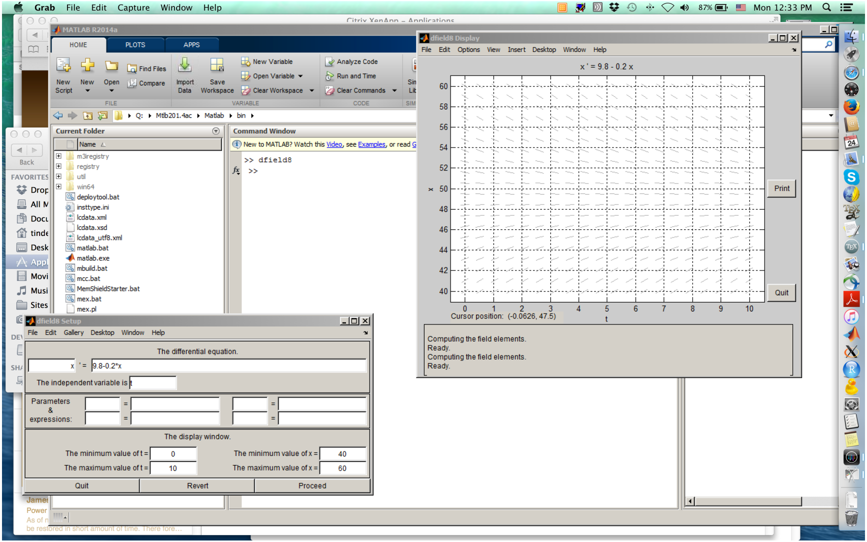
1. Log on `https://goremote.itap.purdue.edu/Citrix/XenApp/auth/login.aspx`
2. Choose
   - Course Software
   - Science
   - Math
   - Dfield

Matlab dfield8 function (2)

Tip: Use Matlab 2014



Matlab dfield8 function (3)



Outline

Mice and owl equation

Specific form:

$$\frac{dp}{dt} = 0.5p - 450 \tag{3}$$

Integration of the equation: We have

$$\frac{p'}{p-900} = \frac{1}{2}$$

Integrating we obtain:

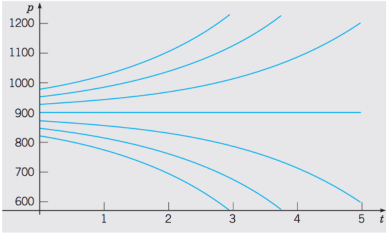
$$p(t) = 900 + c\exp\left(\frac{t}{2}\right), \quad \text{with} \quad c \in \mathbb{R}.$$

Initial data

Family of solutions:

- We have seen: solutions depend on parameter c
- One way to find c: specify value of $p(0)$
- Example: if $p(0) = 850$, then $p(t) = 900 - 50\exp(t/2)$

Graph of solutions according to initial condition:



General solution

Proposition 1.

Equation considered:

$$\frac{dy}{dt} = ay - b, \quad \text{and} \quad y(0) = y_0. \tag{4}$$

Hypothesis:

$$a, b \in \mathbb{R}, \qquad a \neq 0, \qquad y(0) \in \mathbb{R}.$$

Then the unique solution to (4) is given by:

$$y(t) = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}.$$

Mice and owl reloaded

Equation:

$$\frac{dp}{dt} = rp - k$$

Expression of solution: with initial condition $p_0 > 0$,

$$p(t) = \frac{k}{r} + \left[p_0 - \frac{k}{r}\right] e^{rt}$$

Remarks:

- If $p_0 = \frac{k}{r}$, solution stays at equilibrium
- If $p_0 < \frac{k}{r}$, solution decreases until extinction
  $\hookrightarrow$ Negative values of p are physically meaningless
- If $p_0 > \frac{k}{r}$, solution grows exponentially (critics to model?)
- This could be seen on the previous graph

Gravity reloaded

Equation:

$$\frac{dv}{dt} = g - \frac{\gamma}{m} v$$

Expression of solution: with initial condition $v_0 \in \mathbb{R}$,

$$v(t) = \frac{mg}{\gamma} + \left[v_0 - \frac{mg}{\gamma} \right] e^{-\frac{\gamma t}{m}}$$

Remarks:

- If $v_0 = \frac{mg}{\gamma}$, solution stays at equilibrium
- If $v_0 \neq \frac{mg}{\gamma}$, convergence to equilibrium
  $\hookrightarrow$ exponential convergence, rate $\frac{\gamma}{m}$
- From v, one can retrieve position x
  $\hookrightarrow$ find velocity v when a dropped object hits the ground

Outline

Ordinary vs partial differential equations

Ordinary differential equation: depends on one variable only

- Gravity, v= v(t) ; Mice an owl, p=p(t)
- Capacitor with capacitance C, resistance R, inductance L:

$$L\frac{d^2Q}{dt} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

Partial differential equation: depends on two or more variables

- Heat equation:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

- Wave equation:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

Systems of differential equations

Definition: Systems of differential equations
$\hookrightarrow$ when 2 or more unknown functions are involved

Example: Lotka-Volterra predator-pray model

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy \\ \frac{dy}{dt} = -cy + \gamma xy \end{cases}$$

Remark: In many engineering situations
$\hookrightarrow$ lots of coupled differential equations

Order of a differential equation

Definition: Order of a differential equation
= Order of highest derivative appearing in equation

Examples:

- Gravity, Mice-owl: first order
- Capacitor: second order
- Heat, wave: second order partial differential equations

General form of n-th order differential equation:

$$F(y, y', \ldots, y^{(n)}) = 0 \tag{5}$$

Linear and nonlinear equations

Definition: In equation (5),

- If F is linear, differential equation is linear
- If F is not linear, differential equation is nonlinear

Examples:

- Gravity, Mice-owl, Capacitor: linear differential equations
- Heat, wave: linear partial differential equations
- Lotka-Volterra: nonlinear, because of term xy

Remark:

Nonlinear equations are harder to solve than linear equations

Solutions to differential equations

Definition: Solution to equation (5) on $[a, b]$
$\hookrightarrow$ any function ϕ such that $\phi, \phi', \ldots, \phi^{(n)}$ exist and

$$F(\phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) = 0, \quad \text{for} \quad t \in [a, b]$$

Remark: If we have an intuition for a solution to (5)
$\hookrightarrow$ verification is easy

Example: For equation

$$y'' + y = 0,$$

easy to check that $\sin(t)$ and $\cos(t)$ are solutions

Issues related to differential equations

General form of equation:

$$F(y, y', \ldots, y^{(n)}) = 0$$

List of problems:

1. Existence to solution
2. Uniqueness of solution
3. Find exact solutions in simple cases
4. Approximation of solution in complex cases
5. Combine analytic, graphical and numerical methods
   $\hookrightarrow$ to understand solutions