Differential equations: introduction

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Differential equations - MA 26600

Taken from *Elementary differential equations* by Boyce and DiPrima
Outline

1. Basic mathematical models
   - Gravity example
   - Mice and owl example
   - Direction fields

2. Solving affine equations

3. Classification of differential equations
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3. Classification of differential equations
Interest of differential equations

**Differential equation:**
Equation in which the derivative of a function appears.

**Features of differential equations:**

- Theoretical interest
- Always related to a physical system:
  - Fluid dynamics
  - Electrical circuits
  - Population dynamics
  - Economy, finance
- More than 300 years of study
- Still active domain of research
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Expression of Newton’s law

Example of physical situation:
Object falling in the atmosphere near sea level

Notation:
- \( t \) = time variable, in seconds
- \( v \) = velocity, depends on time \( v = v(t) \), in \( m s^{-1} \)
- \( F \) = force
- \( a \) = acceleration

Orientation: Downwards

Newton’s law:

\[
F = m a = m \frac{dv}{dt}
\]
Gravity example (2)

Forces acting on the object:

- Gravity: $mg$, where $g = 9.81 \text{ms}^{-2}$ close to earth
- Air resistance, drag: $-\gamma v$, where $\gamma$ object dependent

Total force: $F = mg - \gamma v$

Resulting equation:

$$m \frac{dv}{dt} = mg - \gamma v$$  (1)
Qualitative study

Specific values for coefficients:
\[ \text{We take } m = 10\text{kg and } \gamma = 2\text{kg s}^{-1} \]

Specific equation:
\[
\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{2}
\]

Note:
- One can solve equation (2)
- Qualitative study: draw conclusions from equation itself

Example of slope:
\[ \text{If } v = 40, \text{ then } \frac{dv}{dt} = 1.8 \]
Direction field

Meaning of the graph:

$\frac{dv}{dt}$ according to values of $v$

What can be seen on the graph:

- Critical value: $v_c = 49\text{ms}^{-1}$, solution to $9.8 - \frac{v}{5} = 0$
- If $v < v_c$: positive slope
- If $v > v_c$: negative slope
Qualitative study (2)

**Equilibrium:** According to the graph
- \( v(t) \equiv v_c \) is solution to (2)
- All solutions converge to \( v_c \) as \( t \rightarrow \infty \)

**Remark:**
1. The facts above will be shown later on
2. \( v_c \) is called *equilibrium* for system (2)

**Generalization:** For general system (1):
- Equilibrium: \( v_c = \frac{mg}{\gamma} \)
- Convergence to equilibrium
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Predator-pray model

Situation:
- $p = p(t)$ = mice population
- Reproduction rate for mice: $r$ mice/month
- Presence of owl: $k$ mice eaten per month

Resulting equation:
$$\frac{dp}{dt} = rp - k$$
Direction field

Meaning of the graph:

Values of $\frac{dv}{dt}$ according to values of $v$

What can be seen on the graph:

- Critical value: $p_c = \frac{k}{r}$, solution to $rp - k = 0$
- If $p < p_c$: negative slope
- If $p > p_c$: positive slope
Qualitative study (2)

Equilibrium: According to the graph
- $p(t) \equiv p_c$ is solution to (2)
- A solution will never converge to $p_c$ as $t \to \infty$
- If $p(0) > p_c$, population increases
- If $p(0) < p_c$, extinction

Remark:
- $p_c$ is an **unstable equilibrium** for system (2)
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Importance of direction fields

General form of an equation:

\[ \frac{dy}{dt} = f(t, y) \]

Conclusions from previous examples:

1. Importance of direction fields graphs \((t, y) \mapsto f(t, y)\)
2. Plotting \((t, y) \mapsto f(t, y)\) is easier than solving the equation
3. It can be done with the help of a computer
Matlab dfield8 function

Remote connexion to Matlab:

2. Choose
   ▶ Course Software
   ▶ Science
   ▶ Math
   ▶ Dfield
Matlab dfield8 function (2)

Tip: Use Matlab 2014
Matlab dfield8 function (3)
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Mice and owl equation

Specific form:

\[
\frac{dp}{dt} = 0.5p - 450
\]  

Integration of the equation: We have

\[
\frac{p'}{p - 900} = \frac{1}{2}
\]

Integrating we obtain:

\[
p(t) = 900 + c \exp \left( \frac{t}{2} \right), \quad \text{with} \quad c \in \mathbb{R}.
\]
Initial data

Family of solutions:
- We have seen: solutions depend on parameter $c$
- One way to find $c$: specify value of $p(0)$
- Example: if $p(0) = 850$, then $p(t) = 900 - 50 \exp(t/2)$

Graph of solutions according to initial condition:
Proposition 1.

Equation considered:

\[
\frac{dy}{dt} = ay - b, \quad \text{and} \quad y(0) = y_0. \tag{4}
\]

Hypothesis:

\[a, b \in \mathbb{R}, \quad a \neq 0, \quad y(0) \in \mathbb{R}.\]

Then the unique solution to (4) is given by:

\[
y(t) = \frac{b}{a} + \left[y_0 - \frac{b}{a}\right] e^{at}.
\]
Mice and owl reloaded

Equation:

$$\frac{dp}{dt} = rp - k$$

Expression of solution: with initial condition $p_0 > 0$,

$$p(t) = \frac{k}{r} + \left[ p_0 - \frac{k}{r} \right] e^{rt}$$

Remarks:

- If $p_0 = \frac{k}{r}$, solution stays at equilibrium
- If $p_0 < \frac{k}{r}$, solution decreases until extinction
  $\leftarrow$ Negative values of $p$ are physically meaningless
- If $p_0 > \frac{k}{r}$, solution grows exponentially (critics to model?)
- This could be seen on the previous graph
Gravity reloaded

Equation:

\[ \frac{dv}{dt} = g - \frac{\gamma}{m} v \]

Expression of solution: with initial condition \( v_0 \in \mathbb{R} \),

\[ v(t) = \frac{mg}{\gamma} + \left[ v_0 - \frac{mg}{\gamma} \right] e^{-\frac{\gamma t}{m}} \]

Remarks:

- If \( v_0 = \frac{mg}{\gamma} \), solution stays at equilibrium
- If \( v_0 \neq \frac{mg}{\gamma} \), convergence to equilibrium
  \( \leftrightarrow \) exponential convergence, rate \( \frac{\gamma}{m} \)
- From \( v \), one can retrieve position \( x \)
  \( \leftrightarrow \) find velocity \( v \) when a dropped object hits the ground
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Ordinary vs partial differential equations

Ordinary differential equation: depends on one variable only
- Gravity, \( v = v(t) \); Mice an owl, \( p = p(t) \)
- Capacitor with capacitance \( C \), resistance \( R \), inductance \( L \):

\[
L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = E
\]

Partial differential equation: depends on two or more variables
- Heat equation:

\[
\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}
\]

- Wave equation:

\[
a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}
\]
Systems of differential equations

Definition: Systems of differential equations
\[ \iff \text{when 2 or more unknown functions are involved} \]

Example: Lotka-Volterra predator-pray model

\[
\begin{align*}
\frac{dx}{dt} &= ax - \alpha xy \\
\frac{dy}{dt} &= -cy + \gamma xy
\end{align*}
\]

Remark: In many engineering situations
\[ \iff \text{lots of coupled differential equations} \]
Order of a differential equation

**Definition:** Order of a differential equation

= Order of highest derivative appearing in equation

**Examples:**
- Gravity, Mice-owl: first order
- Capacitor: second order
- Heat, wave: second order partial differential equations

**General form of \( n \)-th order differential equation:**

\[
F(y, y', \ldots, y^{(n)}) = 0
\]
Linear and nonlinear equations

Definition: In equation (5),
- If $F$ is linear, differential equation is linear
- If $F$ is not linear, differential equation is nonlinear

Examples:
- Gravity, Mice-owl, Capacitor: linear differential equations
- Heat, wave: linear partial differential equations
- Lotka-Volterra: nonlinear, because of term $xy$

Remark:
Nonlinear equations are harder to solve than linear equations
Solutions to differential equations

**Definition:** Solution to equation (5) on \([a, b]\)

\[\rightarrow\] any function \(\phi\) such that \(\phi, \phi', \ldots, \phi^{(n)}\) exist and

\[F(\phi(t), \phi'(t), \ldots, \phi^{(n)}(t)) = 0, \quad \text{for} \quad t \in [a, b]\]

**Remark:** If we have an intuition for a solution to (5)

\[\rightarrow\] verification is easy

**Example:** For equation

\[y'' + y = 0,\]

easy to check that \(\sin(t)\) and \(\cos(t)\) are solutions
Issues related to differential equations

General form of equation:

\[ F(y, y', \ldots, y^{(n)}) = 0 \]

List of problems:

1. Existence to solution
2. Uniqueness of solution
3. Find exact solutions in simple cases
4. Approximation of solution in complex cases
5. Combine analytic, graphical and numerical methods \( \rightarrow \) to understand solutions