Higher order linear equations

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Differential equations - MA 26600

Taken from *Elementary differential equations* by Boyce and DiPrima

Outline

1. General theory
2. Homogeneous equations
3. Method of undetermined coefficients

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General form of nth order linear equation

General form:

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

Initial condition:

- Given by:

$$y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \cdots \quad , \quad y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

- Necessity of n conditions because n integrations performed.

Existence and uniqueness theorem

Theorem 1.

General linear equation:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t). \quad (1)$$

Initial condition:

$$y(t_0) = y_0, \quad y'(t_0) = y_0', \quad \cdots \quad , \quad y^{(n-1)}(t_0) = y_0^{(n-1)}. \quad (2)$$

Hypothesis:

- $t_0 \in I$, where $I = (\alpha, \beta)$.
- $p_1, \ldots, p_n$ continuous on I.

Conclusion:

There exists a unique function y satisfying (1)-(2) on I.

Wronskian of homogeneous equation

Definition 2.

Consider:

- Equation
  $y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$
- n solutions $y_1, y_2, \ldots, y_n$ on interval I.
- $t_0 \in I$.

The Wronskian $W = W[y_1, \ldots, y_n](t_0)$ for $y_1, \ldots, y_n$ at t_0 is:

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) & \ldots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \ldots & y_n'(t_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \ldots & y_n^{(n-1)}(t_0) \end{vmatrix}.$$

Wronskian and determination of solutions

Theorem 3.

Equation:

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0. \qquad (3)$$

Hypothesis:

- Existence of n solutions $y_1, \ldots, y_n$.
- Initial condition $y(t_0) = y_0, \ldots, y^{(n-1)}(t_0)$ assigned.

Conclusion: One can find $c_1, \ldots, c_n$ such that

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

satisfies (3) with initial condition iff

$$W[y_1, \ldots, y_n](t_0) \neq 0$$

Wronskian and uniqueness of solutions

Theorem 4.

Equation: back to (3) that is

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = 0.$$

Hypothesis:

- Existence of n solutions $y_1, \ldots, y_n$.

Conclusion: The general solution

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n, \quad \text{with} \quad c_1, \ldots, c_n \in \mathbb{R}$$

includes all solutions to (3) iff:

$$\text{there exists } t_0 \in I \text{ such that } W[y_1, \ldots, y_n](t_0) \neq 0.$$

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Setting

Equation considered: for $a_0, \ldots, a_n \in \mathbb{R}$,

$$a_0 y^{(n)} + \cdots + a_n y = 0. \tag{4}$$

Characteristic polynomial:

$$Z(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n.$$

Facts about Z:

1. Z has n roots (real, complex or repeated) $r_1, \ldots, r_n$.
2. Z factorizes as: $Z(r) = a_0(r - r_1) \cdots (r - r_n)$.

Construction of solutions

Equation: Homogeneous with constant coefficients (4).

Roots of characteristic polynomial: $r_1, \ldots, r_n$.

Rules to find solutions: separate 3 cases,

1. If $r_j \in \mathbb{R}$ non repeated root,
$$\exp(r_j t) \quad \text{solution to equation (4).}$$

2. If $r_j = \lambda + \imath\mu$ and $r_{j+1} = \lambda - \imath\mu$ conjugate complex roots,
$$\exp(\lambda t)\cos(\mu t),\ \exp(\lambda t)\sin(\mu t) \quad \text{solutions to equation (4).}$$

3. If $r_j \in \mathbb{R}$ repeated root of order s,
$$\exp(r_j t), t\exp(r_j t), \ldots, t^{s-1}\exp(r_j t) \quad \text{solutions to equation (4).}$$

Example of application

Equation:

$$y^{(4)} + y^{(3)} - 7y^{(2)} - y' + 6y = 0. \tag{5}$$

Characteristic polynomial:

$$Z(r) = r^4 + r^3 - 7r^2 - r + 6 = 0.$$

Method to find roots: for roots of Z of the form $\frac{p}{q} \in \mathbb{Q}$:

- p is a factor of a_n.
- q is a factor of a_0.

Application: We seek

- $p \in \{\pm 1, \pm 2, \pm 3, \pm 6\}$ and $q \in \{\pm 1\}$.

Example of application (2)

Roots of Z: We find

$$r_1 = 1, \qquad r_2 = -1, \qquad r_3 = 2, \qquad r_4 = -3.$$

General solution to (5):

$$y = c_1 e^{t} + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-3t}$$

Example of application (3)

Example of initial condition:

$$y(0) = 1, \qquad y'(0) = 0, \qquad y''(0) = -2, \qquad y^{(3)}(0) = -1.$$

Related system:

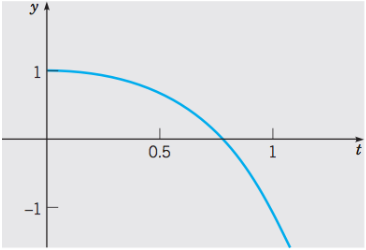
$$\begin{cases} c_1 + c_2 + c_3 + c_4 & = 1 \\ c_1 - c_2 + 2c_3 - 3c_4 & = 0 \\ c_1 + c_2 + 4c_3 + 9c_4 & = -2 \\ c_1 - c_2 + 8c_3 - 27c_4 & = -1 \end{cases}$$

Example of application (4)

Solution to initial value problem:

$$y = \frac{11}{8}e^{t} + \frac{5}{12}e^{-t} - \frac{2}{3}e^{2t} - \frac{1}{8}e^{-3t}$$

Graph of solution:



Example with complex roots

Equation:

$$y^{(4)} - y = 0. \tag{6}$$

Characteristic polynomial:

$$r^4 - 1 = 0.$$

Roots of Z: We find

$$r_1 = 1, \qquad r_2 = \imath, \qquad r_3 = -1, \qquad r_4 = -\imath.$$

General solution to (6):

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos(t) + c_4 \sin(t)$$

Example with complex roots (2)

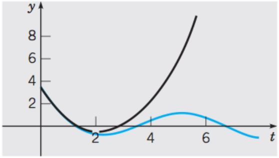
Example of initial condition:

$$y(0) = \frac{7}{2}, \qquad y'(0) = -4, \qquad y''(0) = \frac{5}{2}, \qquad y^{(3)}(0) = -2.$$

Solution to initial value problem:

$$y = 3e^{-t} + \frac{1}{2}\cos(t) - \sin(t)$$

Graph of solution: with 2 slightly $\neq$ initial conditions



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Method of undetermined coefficients

Nonhomogeneous linear equation with constant coefficients:

$$a_0 y^{(n)} + \cdots + a_n y = g(t).$$

Aim: Find a particular solution Y to the equation.

Table of possible guess: restricted to a limited number of cases,

Function g	Guess
$\alpha \exp(at)$	$A \exp(at)$
$\alpha \sin(\omega t) + \beta \cos(\omega t)$	$A \sin(\omega t) + B \cos(\omega t)$
$\alpha_n t^n + \cdots + \alpha_0$	$A_n t^n + \cdots + A_0$
$(\alpha_n t^n + \cdots + \alpha_0) \exp(at)$	$(A_n t^n + \cdots + A_0) \exp(at)$
$(\alpha \sin(\omega t) + \beta \cos(\omega t)) \exp(at)$	$(A \sin(\omega t) + B \cos(\omega t)) \exp(at)$

Elaboration of the guess

Situation:

- Equation: $a_0 y^{(n)} + \cdots + a_n y = g(t)$
- g solution to homogeneous equation
  $\Longrightarrow g = c \exp(rt)$, where r root of Z.
- Let $s \equiv$ multiplicity of r.

Particular solution: of the form

$$Y(t) = t^s \exp(rt).$$

Example of application

Equation:

$$y^{(3)} - 3y^{(2)} + 3y' - y = 4e^t.$$

Characteristic polynomial:

$$Z(r) = (r-1)^3.$$

Solution to homogeneous equation: for $c_1, c_2, c_3 \in \mathbb{R}$,

$$y_c = c_1 e^t + c_2 t e^t + c_3 t^2 e^t.$$

Example of application (2)

Guess for particular solution:

$$Y(t) = At^3 e^t$$

General solution:

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + \frac{2}{3} t^3 e^t.$$

Example with superposition

Equation:

$$y^{(3)} - 4y' = t + 3\cos(t) + e^{-2t}.$$

Characteristic polynomial:

$$Z(r) = r(r-2)(r+2).$$

Solution to homogeneous equation: for $c_1, c_2, c_3 \in \mathbb{R}$,

$$y_c = c_1 + c_2 e^{2t} + c_3 e^{-2t}.$$

Example with superposition (2)

Sub-equation 1:

$$y^{(3)} - 4y' = t.$$

Guess for particular solution 1:

$$Y_1(t) = t(A_0 t + A_1) \quad \Longrightarrow \quad A_0 = -\frac{1}{8},\ A_1 = 0.$$

Sub-equation 2:

$$y^{(3)} - 4y' = \cos(t).$$

Guess for particular solution 2:

$$Y_2(t) = B\cos(t) + C\sin(t) \quad \Longrightarrow \quad B = 0,\ C = -\frac{3}{5}.$$

Example with superposition (3)

Sub-equation 3:

$$y^{(3)} - 4y' = e^{-2t}.$$

Guess for particular solution 3:

$$Y_3(t) = Dte^{-2t} \quad \Longrightarrow \quad D = \frac{1}{8}.$$

General solution:

$$y = c_1 + c_2 e^{2t} + c_3 e^{-2t} - \frac{t^2}{8} - \frac{3}{5}\sin(t) + \frac{t}{8}e^{-2t}.$$