# Review problems

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Differential equations - MA 266

Taken from *Elementary differential equations* by Boyce and DiPrima

# Integrating factor

Problem: Let $y$ be the solution of:

$$t^2 y' + 3ty = 6t, \qquad y(1) = 0.$$

Find $y'(1)$.

Answer:

$$y(t) = 2 - \frac{2}{t^3}, \qquad y'(1) = 6$$

# Separable equation

Problem 2.2.18:

Find the solution to the initial value problem:

$$y' = \frac{e^{-x} - e^{x}}{3 + 4y}, \qquad y(0) = 1$$

Answer:

$$3y + 2y^2 + e^{-x} + e^{x} = 7$$

# Homogeneous equation

Problem 2.2.34: Solve the following differential equation on $(0, \infty)$.

$$y' = \frac{x}{2y} + \frac{3y}{2x}, \qquad y(1) = 2$$

Answer:

$$y^2 = cx^3 - x^2$$

# Tank

Problem 2.3.4: A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the equation corresponding to this situation and solve it.

Answer:

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200 + t}$$

# Equilibrium for autonomous equations

Problems 2.5.8 - 2.5.12: Classify the equilibrium points and sketch some graphs of solutions for the following equations, for which $y_0 \in \mathbb{R}$:

1. $y' = -k(y-1)^2$, with $k > 0$
2. $y' = y^2(4 - y^2)$

Answer:

1. 1 semi-stable equilibrium
2. -2 unstable, 0 semi-stable, 2 stable

# Exact equation

Problem: Solve the following equation:

$$(2xy^2 + 2y) + (2x^2y + 2x)y' = 0, \qquad y(1) = 1$$

Answer:

$$x^2y^2 + 2xy = 3$$

# Euler's method

Problem 2.7.1: Use the Euler method, with time step $h = 0.5$, in order to find an approximate value of $y(1)$ for the following equation:

$$y' = 3 + t - y, \qquad y(0) = 1.$$

Answer:

$$\hat{y}(1) = 2.75$$

# Undamped vibration

Problem: Find the solution of the equation:

$$y'' + 4y = 0, \qquad y(0) = 2, \qquad y'(0) = 4,$$

under the form $y = R\cos(\omega t - \delta)$.

Answer:

$$y = 2\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$$

# Undetermined coefficients

Problem: Consider the equation

$$y^{(4)} - 2y^{(3)} - 3y^{(2)} = 4t^2 - 3 + e^{3t} + e^{t} - 6\sin(t).$$

Find the form of a general solution for this system.

Answer: the particular solution is of the form

$$Y = d_1 t^4 + d_2 t^3 + d_3 t^2 + d_4 t e^{3t} + d_5 e^{-t} + d_6 \cos(t) + d_7 \sin(t).$$

The general solution is of the form:

$$y = c_1 e^{-t} + c_2 e^{3t} + c_3 t + c_4 + Y$$

# Variation of parameters

Problem: Consider the following equation:

$$y'' + 2y' + y = \frac{e^{-t}}{t}.$$

Find a particular solution $Y$ of the non homogeneous problem, under the form $Y = u_1 y_1 + u_2 y_2$ where $y_1, y_2$ are solutions of the homogeneous problem.

Answer:

$$Y = t\ln(t)e^{-t}$$

# Interval of definition

Problem: Find the maximal interval of definition for

$$\begin{cases} (4-t^2)\, y^{(3)} + 2ty = 3\ln(t) \\ y(1) = -3, \quad y'(1) = \pi, \quad y''(1) = 0. \end{cases}$$

Answer:

$$I = (0, 2)$$

# Reduction of order

Problem: A solution of the equation

$$(x-1)y'' - xy' + y, \qquad x > 1,$$

is given by $y_1 = e^x$. Find a second fundamental solution by the reduction of order method.

Answer:

$$y_2 = x$$

# Steady state

Problem: Consider the equation

$$y'' + 4y' + 5y = 2\cos(t) - \sin(t),$$

Find the steady state of the system.

Answer:

$$Y = \frac{1}{8}\cos(t) + \frac{3}{8}\sin(t).$$

# Laplace transform

Problem: Find the Laplace transform of the following function:

$$h(t) = \begin{cases} 0, & 0 \leq t < 3 \\ (t-1)e^{2t} & t \geq 3 \end{cases}$$

Answer:

$$H(s) = \frac{2s-1}{(s-2)^2}\, e^{-3s+6}.$$

# Inverse Laplace transform

Problem: Find the inverse Laplace transform of the following function:

$$F(s) = \frac{s^2 - 4s + 12}{(s+1)(s^2 - 6s + 10)}.$$

Answer:

$$f(t) = e^{-t} + 2e^{3t}\sin(t).$$

# Equation with impulse

Problem: Consider the equation

$$y'' + 9y = \delta(t-2), \qquad y(0) = 0, \; y'(0) = 0.$$

Find the expression of $y$.

Answer:

$$y(t) = \frac{1}{3} u_2(t) \sin(3(t-2))$$

# Convolution integral

Problem: Find the Laplace transform of the following function:

$$h(t) = e^t \int_0^t \exp\left(-(t-\tau)\right) \cos(2\tau)\, d\tau$$

Answer:

$$H(s) = \frac{s-1}{s(s^2-2s+5)},$$

# System with real eigenvalues

Problem 7.5.16: Solve the initial problem

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Answer:

$$y(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}.$$

# Phase portrait

Problem 7.6.3: Describe the phase portrait of the system

$$\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}.$$

Answer:

$\mathbf{0}$ is a center, counterclockwise motion.

# System with complex eigenvalues

Problem 7.6.9: Solve the initial problem

$$\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \qquad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Describe its phase diagram.

Answer:

$$y(t) = \begin{pmatrix} \cos(t) - 3\sin(t) \\ \cos(t) - \sin(t) \end{pmatrix} e^{-t}.$$

# Nonhomogeneous system

Problem: Find a particular solution for the following equation:

$$\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

Answer:

$$\mathbf{x}(t) = \frac{1}{2} \begin{pmatrix} -t + \frac{1}{2} \\ -t + \frac{3}{2} \end{pmatrix} e^t.$$

# Final message for the final

1. **WORK HARD**
2. **GOOD LUCK!!**