Axioms of Probability

Samy Tindel

Purdue University

Probability - MA 416

Mostly taken from *A first course in probability* by S. Ross

Outline

Outline

Global objective

Aim: Introduce

- Sample space
- Events of an experiment
- Probability of an event
- Show how probabilities can be computed in certain situations

Outline

Sample space

Situation: We run an experiment for which

- Specific outcome is unknown
- Set S of possible outcomes is known

Terminology:

In the context above S is called sample space

Examples of sample spaces

Tossing two dice: We have

$$\begin{aligned} S &= \{1,2,3,4,5,6\}^2 \\ &= \{(i,j);\ i,j = 1,2,3,4,5,6\} \end{aligned}$$

Lifetime of a transistor: We have

$$S = \mathbb{R}_+ = \{x \in \mathbb{R};\ 0 \le x < \infty\}$$

Events

Definition 1.

Consider

- Experiment with sample space S
- A subset E of S

Then

E is called event

Example of event (1)

Tossing two dice: We have

$$S = \{1,2,3,4,5,6\}^2$$

Event: We define

$$E = (\text{Sum of dice is equal to 7})$$

Example of event (2)

Description of E as a subset:

$$E = \{(1,6); (2,5); (3,4); (4,3); (5,2); (6,1)\} \subset S$$

Second example of event (1)

Lifetime of a transistor: We have

$$S = \mathbb{R}_+ = \{x \in \mathbb{R};\, 0 \le x < \infty\}$$

Event: We define

$$E = (\text{Transistor does not last longer than 5 hours})$$

Second example of event (2)

Description of E as a subset:

$$E = [0, 5] \subset S$$

Operations on events

Complement: E^c is the set of elements of S not in E

Two dice example:

$$E^c = \text{"Sum of two dice different from 7"}$$

Union, Intersection: For the two dice example, if

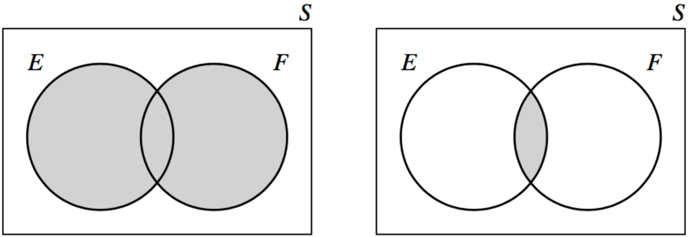
$$B = \text{"Sum of two dice is divisible by 3"}$$
$$C = \text{"Sum of two dice is divisible by 4"}$$

Then

$$B \cup C = \text{"Sum of two dice is divisible by 3 or 4"}$$
$$B \cap C = BC = \text{"Sum of two dice is divisible by 3 and 4"}$$

Illustration (1)

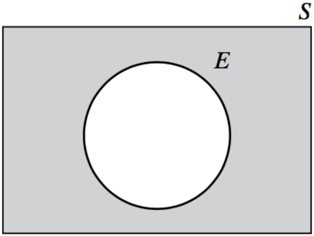
Union and intersection:



(a) Shaded region: $E \cup F$. (b) Shaded region: EF.

Illustration (2)

Complement:



(c) Shaded region: E^c.

Illustration (3)

Subset:

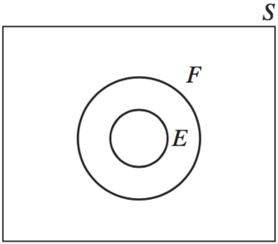


Figure: $E \subset F$

Laws for elementary operations

Commutative law:

$$E \cup F = F \cup E, \qquad EF = FE$$

Associative law:

$$(E \cup F) \cup G = E \cup (F \cup G), \qquad E(FG) = (EF)G$$

Distributive laws:

$$\begin{aligned} (E \cup F)G &= EG \cup EG \\ (EF) \cup G &= (E \cup G)(F \cup G) \end{aligned}$$

Illustration

Distributive law:

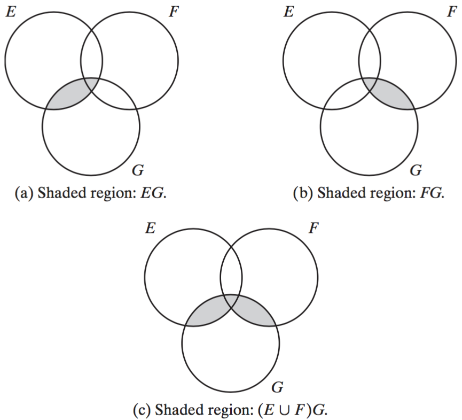


Figure: $(E \cup F)G = EG \cup FG$

De Morgan's laws

Proposition 2.

Let

- S sample space
- $E_1, \ldots, E_n$ events

Then

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c$$

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

Proof (1)

Proof of $(\cup_{i=1}^{n} E_i)^c \subset \cap_{i=1}^{n} E_i^c$:

Assume $x \in (\cup_{i=1}^{n} E_i)^c$ Then

$$\begin{aligned} x \notin \cup_{i=1}^{n} E_i &\implies \text{for all } i \leq n,\ x \notin E_i \\ &\implies \text{for all } i \leq n,\ x \in E_i^c \\ &\implies x \in \cap_{i=1}^{n} E_i^c \end{aligned}$$

Proof (2)

Proof of $\cap_{i=1}^n E_i^c \subset (\cup_{i=1}^n E_i)^c$:

Assume $x \in \cap_{i=1}^n E_i^c$ Then

$$\begin{aligned} \text{for all } i \leq n,\, x \in E_i^c &\implies \text{for all } i \leq n,\, x \notin E_i \\ &\implies x \notin \cup_{i=1}^n E_i \\ &\implies x \in (\cup_{i=1}^n E_i)^c \end{aligned}$$

Outline

Definition of probability

Definition 3.

A probability is an application which assigns a number (chances to occur) to any event E. It must satisfy 3 axioms

1. $$0 \leq \mathbf{P}(E) \leq 1$$
2. $$\mathbf{P}(S) = 1$$
3. If $E_i E_j = \varnothing$ for $i, j \geq 1$ such that $i \neq j$, then
$$\mathbf{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbf{P}(E_i)$$

Easy consequence of the axioms

Proposition 4.

Let **P** be a probability on S. Then

1. $$\mathbf{P}(\emptyset) = 0$$
2. For $n \geq 1$,
   if $E_i E_j = \emptyset$ for $1 \leq i, j \leq n$ such that $i \neq j$ then
   $$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mathbf{P}(E_i)$$

Example: dice tossing

Experiment: tossing one dice

Model: $S = \{1, \ldots, 6\}$ and

$$\mathbf{P}(\{s\}) = \frac{1}{6}, \quad \text{for all} \quad s \in S$$

Probability of an event: If $E =$ "even number obtained", then

$$\begin{aligned} \mathbf{P}(E) &= \mathbf{P}(\{2,4,6\}) = \mathbf{P}(\{2\} \cup \{4\} \cup \{6\}) \\ &= \mathbf{P}(\{2\}) + \mathbf{P}(\{4\}) + \mathbf{P}(\{6\}) = \frac{3}{6} = \frac{1}{2} \end{aligned}$$

Outline

Probability of a complement

Proposition 5.

Let

- $\mathbf{P}$ a probability on a sample space S
- E an event

Then

$$\mathbf{P}\left(E^c\right) = 1 - \mathbf{P}(E)$$

Proof

Use Axioms 2 and 3:

$$1 = \mathbf{P}(S) = \mathbf{P}\left(E \cup E^c\right) = \mathbf{P}\left(E\right) + \mathbf{P}\left(E^c\right)$$

Probability of a subset

Proposition 6.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events, such that $E \subset F$

Then

$$\mathbf{P}(E) \leq \mathbf{P}(F)$$

Proof

Decomposition of F: Write

$$F = E \cup E^c F$$

Use Axioms 1 and 3: Since E and $E^c F$ are disjoint,

$$\mathbf{P}(F) = \mathbf{P}\left(E \cup E^c F\right) = \mathbf{P}\left(E\right) + \mathbf{P}\left(E^c F\right) \geq \mathbf{P}(E)$$

Probability of a non disjoint union

Proposition 7.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events

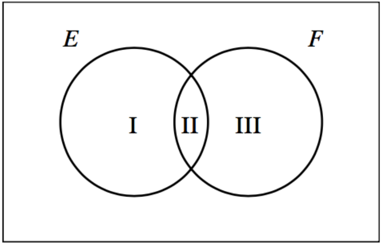
Then

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$

Proof

Decomposition of $E \cup F$:

$$E \cup F = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$$



Proof (2)

Decomposition for probabilities: We have

$$\begin{aligned}\mathbf{P}(E \cup F) &= \mathbf{P}(\mathrm{I}) + \mathbf{P}(\mathrm{II}) + \mathbf{P}(\mathrm{III}) \\ \mathbf{P}(E) &= \mathbf{P}(\mathrm{I}) + \mathbf{P}(\mathrm{II}) \\ \mathbf{P}(F) &= \mathbf{P}(\mathrm{II}) + \mathbf{P}(\mathrm{III})\end{aligned}$$

Conclusion: Since $\mathrm{II} = E \cap F$, we get

$$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(\mathrm{II}) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$$

Application of Propositions 5 and 7

Experiment: dice tossing
$\hookrightarrow S = \{1, \ldots, 6\}$ and $\mathbf{P}(\{s\}) = \frac{1}{6}$ for all $s \in S$

Events: We consider the 2 events

$$A = \text{"even outcome"}$$
$$B = \text{"outcome multiple of 3"}$$

Application of Propositions 5 and 7 (Ctd)

Experiment: dice tossing
$\hookrightarrow S = \{1, \ldots, 6\}$ and $\mathbf{P}(\{s\}) = \frac{1}{6}$ for all $s \in S$

Events:
We consider $A =$ "even outcome" and $B =$ "outcome multiple of 3"
$\Rightarrow A = \{2, 4, 6\}$ and $B = \{3, 6\}$
$\Rightarrow \mathbf{P}(A) = 1/2$ and $\mathbf{P}(B) = 1/3$

Applying Propositions 5 and 7:
$\mathbf{P}(A^c) = 1 - \mathbf{P}(A) = 1/2$
$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1/2 + 1/3 - \mathbf{P}(\{6\}) = 2/3$

Verification:
$A^c = \{1, 3, 5\} \Rightarrow \mathbf{P}(A^c) = 1/2$
$A \cup B = \{2, 3, 4, 6\} \Rightarrow \mathbf{P}(A \cup B) = 4/6 = 2/3$

Inclusion-exclusion identity

Proposition 8.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $E_1, \ldots, E_n$

Then

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \leq i_1 < \cdots < i_r \leq n} \mathbf{P}\left(E_{i_1} \cdots E_{i_r}\right)$$

Proof for $n = 3$

Apply Proposition 7:

$$\begin{aligned}\mathbf{P}(E_1 \cup E_2 \cup E_3) &= \mathbf{P}(E_1 \cup E_2) + \mathbf{P}(E_3) - \mathbf{P}((E_1 \cup E_2)E_3) \\ &= \mathbf{P}(E_1 \cup E_2) + \mathbf{P}(E_3) - \mathbf{P}(E_1E_3 \cup E_2E_3)\end{aligned}$$

Apply Proposition 7 to $E_1 \cup E_2$ and $E_1E_3 \cup E_2E_3$:

$$\mathbf{P}(E_1 \cup E_2 \cup E_3) = \sum_{1 \le i_1 \le 3} \mathbf{P}(E_{i_1}) - \sum_{1 \le i_1 < i_2 \le 3} \mathbf{P}(E_{i_1}E_{i_2}) + \mathbf{P}(E_1E_2E_3)$$

Case of general n: By induction

Bounds for $\mathbf{P}(\cup_{i=1}^{n} E_i)$

Proposition 9.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $E_1, \ldots, E_n$

Then

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{1\leq i\leq n} \mathbf{P}\left(E_i\right)$$

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right) \geq \sum_{1\leq i\leq n} \mathbf{P}\left(E_i\right) - \sum_{1\leq i_1<i_2\leq n} \mathbf{P}\left(E_{i_1} E_{i_2}\right)$$

Bounds for $\mathbf{P}(\cup_{i=1}^{n} E_i)$ – Ctd

Proposition 10.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $E_1, \ldots, E_n$

Then

$$\mathbf{P}\left(\bigcup_{i=1}^{n} E_i\right)$$
$$\leq \sum_{1 \leq i \leq n} \mathbf{P}\left(E_i\right) - \sum_{1 \leq i_1 < i_2 \leq n} \mathbf{P}\left(E_{i_1} E_{i_2}\right) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \mathbf{P}\left(E_{i_1} E_{i_2} E_{i_3}\right)$$

Proof

Notation: Set

$$B_i = E_1^c \cdots E_{i-1}^c$$

Identity:

$$\mathbf{P}\left(\cup_{i=1}^n E_i\right) = \mathbf{P}(E_1) + \sum_{i=2}^n \mathbf{P}\left(B_i E_i\right)$$

Second identity: Since $B_i = (\cup_{j<i} E_j)^c$,

$$\mathbf{P}\left(B_i E_i\right) = \mathbf{P}\left(E_i\right) - \mathbf{P}\left(\cup_{j<i} E_j E_i\right)$$

Partial conclusion:

$$\mathbf{P}\left(\cup_{i=1}^n E_i\right) = \sum_{1\le i\le n} \mathbf{P}(E_i) - \sum_{1\le i\le n} \mathbf{P}\left(\cup_{j<i} E_j E_i\right)$$

Proof (2)

Recall:

$$\mathbf{P}\left(\cup_{i=1}^{n} E_i\right) = \sum_{1 \le i \le n} \mathbf{P}(E_i) - \sum_{1 \le i \le n} \mathbf{P}\left(\cup_{j<i} E_j E_i\right) \tag{1}$$

Direct consequence of (1):

$$\mathbf{P}\left(\cup_{i=1}^{n} E_i\right) \le \sum_{1 \le i \le n} \mathbf{P}(E_i) \tag{2}$$

Application of (2) to $\mathbf{P}(\cup_{j<i} E_j E_i)$:

$$\mathbf{P}\left(\cup_{j<i} E_j E_i\right) \le \sum_{j<i} \mathbf{P}\left(E_j E_i\right)$$

Plugging into (1) we get

$$\mathbf{P}\left(\cup_{i=1}^{n} E_i\right) \ge \sum_{1 \le i \le n} \mathbf{P}(E_i) - \sum_{j<i} \mathbf{P}\left(E_j E_i\right)$$

Outline

Model

Hypothesis: We assume

- $S = \{s_1, \dots, s_N\}$ finite.
- $\mathbf{P}(\{s_i\}) = \frac{1}{N}$ for all $1 \le i \le N$

Alert:

This is an important but very particular case of probability space

Example: tossing 4 dice

$\hookrightarrow S = \{1, \dots, 6\}^4$ and

$$\begin{aligned}\mathbf{P}(\{(1,1,1,1)\}) &= \mathbf{P}(\{(1,1,1,2)\}) = \cdots = \mathbf{P}(\{(6,6,6,6)\}) \\ &= \frac{1}{6^4} = \frac{1}{1296}\end{aligned}$$

Computing probabilities

Proposition 11.

Hypothesis: We assume

- $S = \{s_1, \ldots, s_N\}$ finite.
- $\mathbf{P}(\{s_i\}) = \frac{1}{N}$ for all $1 \le i \le N$

In this situation, let $E \subset S$ be an event. Then

$$\mathbf{P}(E) = \frac{\mathrm{Card}(E)}{N} = \frac{|E|}{N} = \frac{\#\text{ outcomes in } E}{\#\text{ outcomes in } S}$$

Example: tossing one dice

Model: tossing one dice, that is

$$S = \{1, \ldots, 6\}, \qquad \mathbf{P}(\{s_i\}) = \frac{1}{6}$$

Computing a simple probability: Let $E =$ "even outcome". Then

$$\mathbf{P}(E) = \frac{|E|}{N} = \frac{3}{6} = \frac{1}{2}$$

Main problem: compute $|E|$ in more complex situations
$\hookrightarrow$ Counting

Example: drawing balls (1)

Situation: We have

- A bowl with 6 White and 5 Black balls
- We draw 3 balls

Problem: Compute

$$\mathbf{P}(E), \quad \text{with} \quad E = \text{"Draw 1 W and 2 B"}$$

Example: drawing balls (2)

Model 1: We take

- $S = \{\text{Ordered triples of balls, tagged from 1 to 11}\}$
- $\mathbf{P}$ = Uniform probability on S

Computing $|S|$: We have

$$|S| = 11 \cdot 10 \cdot 9 = 990$$

Decomposition of E: We have

$$E = \mathrm{WBB} \cup \mathrm{BWB} \cup \mathrm{BBW}$$

Example: drawing balls (3)

Counting E:

$$|E| = |\mathrm{WBB}| + |\mathrm{BWB}| + |\mathrm{BBW}| = 3 \times (6 \times 5 \times 4) = 360$$

Probability of E: We get

$$\mathbf{P}(E) = \frac{|E|}{|S|} = \frac{360}{990} = \frac{4}{11} = 36.4\%$$

Example: drawing balls (4)

Model 2: We take

- $S = \{\text{Non ordered triples of balls, tagged from 1 to 11}\}$
- $\mathbf{P} =$ Uniform probability on S

Computing $|S|$: We have

$$|S| = \binom{11}{3} = 165$$

Decomposition of E: We have

$$E = \{\text{Triples with 2 B and 1 W}\}$$

Example: drawing balls (5)

Counting E:

$$|E| = \binom{5}{2} \times \binom{6}{1} = 60$$

Probability of E: We get

$$\mathbf{P}(E) = \frac{|E|}{|S|} = \frac{60}{165} = \frac{4}{11} = 36.4\%$$

Remark:

When experiment $\equiv$ draw k objects from n objects, two choices:

1. Considered the ordered set of possible draws
2. Consider the draws as unordered

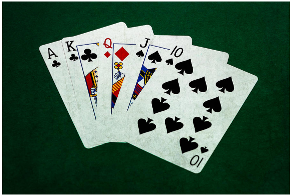
Example: poker game (1)

Situation: Deck of 52 cards and

- Hand: 5 cards
- Straight: distinct consecutive values, not of the same suit

Problem: Compute

$$\mathbf{P}(E), \quad \text{with} \quad E = \text{"Straight is drawn"}$$



Example: poker game (2)

Model: We take

- $S = \{\text{Non ordered hands of cards}\}$
- $\mathbf{P} = $ Uniform probability on S

Computing $|S|$: We have

$$|S| = \binom{52}{5} = 2,598,960$$

Decomposition of E: We have

$$E = \{\text{Straight hands}\}$$

Example: poker game (3)

Counting E: We have

- $\#$ possible 1,2,3,4,5: 4^5
- $\#$ possible 1,2,3,4,5 not of the same suit: $4^5 - 4$
- $\#$ possible values of straights: 10

Thus

$$|E| = 10(4^5 - 4) = 10,200$$

Probability of E: We get

$$\mathbf{P}(E) = \frac{|E|}{|S|} = \frac{10(4^5 - 4)}{\binom{52}{5}} = 0.39\%$$

Example: roommate pairing (1)

Situation: We have

- A football team with 20 Offensive and 20 Defensive players
- Players are paired by 2 for roommates
- Pairing made at random

Problem: Find probability of

1. No offensive-defensive roommate pairs
2. $2i$ offensive-defensive roommate pairs

Example: roommate pairing (2)

Model: We take

- $S = \{\text{Non ordered pairings of 40 players}\}$
- $\mathbf{P} = \text{Uniform probability on } S$

Computing $|S|$: We have

$$|S| = \frac{1}{20!}\binom{40}{2, 2, \ldots, 2} = \frac{40!}{2^{20}\, 20!} \simeq 3.20\ 10^{23}$$

First event E_0: We set

$$E_0 = \{\text{No Offensive-Defensive pairing}\}$$

Example: roommate pairing (3)

Counting E_0: We have

$$|E_0| = (\#\text{ O–O pairings}) \times (\#\text{ D–D pairings})$$
$$= \left(\frac{20!}{2^{10}\,10!}\right)^2$$

Computing $\mathbf{P}(E_0)$:

$$\mathbf{P}(E_0) = \frac{|E_0|}{|S|} = \frac{(20!)^3}{(10!)^2 40!} \simeq 1.34\ 10^{-6}$$

Example: roommate pairing (4)

Events E_{2i}: We set

$$E_{2i} = \{2i \text{ Offensive-Defensive pairings}\}$$

Counting E_{2i}: We have

- # selections of $2i$ O & $2i$ D: $\binom{20}{2i}^2$
- # $2i$ O–D pairings: $(2i)!$
- # $(20-2i)$ O & D intra-pairings: $\left(\frac{(20-2i)!}{2^{10-i}\,(10-i)!}\right)^2$

Thus we get

$$|E_{2i}| = \binom{20}{2i}^2 (2i)! \left(\frac{(20-2i)!}{2^{10-i}\,(10-i)!}\right)^2$$

Example: roommate pairing (5)

Computing $\mathbf{P}(E_{2i})$:

$$\mathbf{P}(E_{2i}) = \frac{|E_{2i}|}{|S|} = \frac{\binom{20}{2i}^2 (2i)! \left(\frac{(20-2i)!}{2^{10-i}\,(10-i)!}\right)^2}{\frac{40!}{2^{20}\,20!}}$$

Some values of $\mathbf{P}(E_{2i})$:

$$\begin{aligned} \mathbf{P}(E_0) &\simeq 1.34\ 10^{-6} \\ \mathbf{P}(E_{10}) &\simeq 0.35 \\ \mathbf{P}(E_{20}) &\simeq 7.6\ 10^{-6} \end{aligned}$$

Example: husband-wife placement (1)

Situation: We have

- A round table
- 10 married couples
- Placement at random

Problem: Find probability that

1. n couples sit next to each other
2. No husband sits next to his wife

Example: husband-wife placement (2)

Model: We take

- $S = \{\text{Permutations of 20 persons}\}/\{\text{Cyclic transformations}\}$
- $\mathbf{P} = $ Uniform probability on S

Computing $|S|$: We have

$$|S| = \frac{20!}{20} = 19!$$

Events E_i: We set

$$E_i = \{i\text{th husband sits next to his wife}\}$$

Example: husband-wife placement (3)

Basic idea: Let $i_1 < \cdots < i_n$. Then on $E_{i_1} \cdots E_{i_n}$

- The n couples $i_1, \ldots, i_n$ are considered as one entity
- We are left with the placement of $20 - n$ entities

Counting $E_{i_1} \cdots E_{i_n}$: We have

- $\#$ placements of $(20 - n)$ entities: $(20 - n - 1)!$
- $\#$ wife-husband placements next to each other: 2^n

Thus

$$|E_{i_1} \cdots E_{i_n}| = 2^n \, (19 - n)!$$

Example: husband-wife placement (4)

Second event, n couples sit together: For $1 \leq n \leq 10$, define

$$\begin{aligned} A_n &= \{n \text{ couples sitting next to each other}\} \\ &= \bigcup_{1 \leq i_1 < \cdots < i_n \leq 10} \left(E_{i_1} \cdots E_{i_n} \right) \end{aligned}$$

Then

$$\begin{aligned} \mathbf{P}(A_n) &= \sum_{1 \leq i_1 < \cdots < i_n \leq 10} \mathbf{P}\left(E_{i_1} \cdots E_{i_n} \right) \\ \mathbf{P}(A_n) &= \binom{10}{n} \frac{2^n (19-n)!}{19!} \end{aligned}$$

Example: husband-wife placement (5)

Third event, no couple sits together: Define

$$A_0 = \{\text{no couple sitting next to each other}\}$$

Then

$$\begin{aligned} A_0^c &= \{\text{at least one couple sitting next to each other}\} \\ &= \bigcup_{i=1}^{10} E_i \end{aligned}$$

Example: husband-wife placement (6)

Computing $\mathbf{P}(A_0^c)$: Thanks to Proposition 8

$$\begin{aligned}
\mathbf{P}\left(A_0^c\right) &= \mathbf{P}\left(\cup_{i=1}^{10} E_i\right) \\
&= \sum_{n=1}^{10} (-1)^{n+1} \sum_{1 \le i_1 < \cdots < i_n \le 10} \mathbf{P}\left(E_{i_1} \cdots E_{i_n}\right) \\
&= \sum_{n=1}^{10} (-1)^{n+1} \binom{10}{n} \frac{2^n (19-n)!}{19!}
\end{aligned}$$

Computing $\mathbf{P}(A_0)$: We get

$$\mathbf{P}\left(A_0\right) = 1 + \sum_{n=1}^{10} (-1)^n \binom{10}{n} \frac{2^n (19-n)!}{19!}$$

Outline

Probabilities for increasing sequences

Proposition 12.

Let

- $\mathbf{P}$ a probability on a sample space S
- An increasing family of events $\{E_i;\, i \geq 1\}$
- Set $\lim_{n\to\infty} E_n = \cup_{i=1}^{\infty} E_i$

Then

$$\mathbf{P}\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mathbf{P}(E_n)$$

Proof (1)

Decomposition with exclusive sets: Define

$$F_n = E_n \, E_{n-1}^c$$

Then the F_i are mutually exclusive and we have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$$

$$\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i$$

Proof (2)

Computation for $\mathbf{P}(\lim_{n\to\infty} E_n)$:

$$\begin{aligned}
\mathbf{P}\left(\lim_{n\to\infty} E_n\right) &= \mathbf{P}\left(\cup_{i=1}^{\infty} E_i\right) \\
&= \mathbf{P}\left(\cup_{i=1}^{\infty} F_i\right) \\
&= \sum_{i=1}^{\infty} \mathbf{P}\left(F_i\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{P}\left(F_i\right) \\
&= \lim_{n\to\infty} \mathbf{P}\left(\cup_{i=1}^{n} F_i\right) \\
&= \lim_{n\to\infty} \mathbf{P}\left(\cup_{i=1}^{n} E_i\right) \\
&= \lim_{n\to\infty} \mathbf{P}\left(E_n\right)
\end{aligned}$$

Probabilities for decreasing sequences

Proposition 13.

Let

- $\mathbf{P}$ a probability on a sample space S
- An decreasing family of events $\{E_i;\, i \geq 1\}$
- Set $\lim_{n\to\infty} E_n = \cap_{i=1}^{\infty} E_i$

Then

$$\mathbf{P}\left(\lim_{n\to\infty} E_n\right) = \lim_{n\to\infty} \mathbf{P}(E_n)$$