Continuous random variables

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Probability - MA 416

Mostly taken from *A first course in probability*
by S. Ross

Outline

Outline

General definition

Definition 1.

Let

- **P** a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$

We say that X is a continuous random variable if
$\hookrightarrow$ There exists $f \geq 0$ such that for "all" $B \subset \mathbb{R}$ we have

$$\mathbf{P}(X \in B) = \int_B f(x)\, dx$$

The function f is called
$\hookrightarrow$ the probability density function of the random variable X

Law of X according to f

Type of information obtained with f: We have

$$
\begin{aligned}
\mathbf{P}(a \leq X \leq b) &= \int_a^b f(x)\,dx \\
\mathbf{P}(X = a) &= 0 \\
F(a) = \mathbf{P}(X \leq a) &= \int_{-\infty}^a f(x)\,dx
\end{aligned}
$$

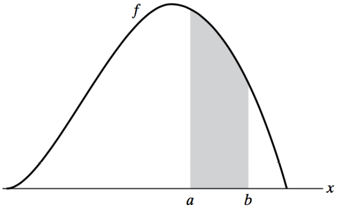


Figure: $\mathbf{P}(a \leq X \leq b) = \int_a^b f(x)\,dx$

Example: radio tube (1)

Situation:

- X = lifetime of a radio tube
- Density of X:

$$f(x) = \frac{100}{x^2} \mathbf{1}_{(100,\infty)}(x)$$

- We have 5 tubes in a set

Question: Probability that 2 of the 5 tubes have to be replaced within the first 150h of operation

Example: radio tube (2)

Family of events: We define

- X_i = lifetime of tube i
- E_i = "tube i has to be replaced within the first 150h of operation

Probability of E_i:

$$\begin{aligned} \mathbf{P}(E_i) &= \mathbf{P}(X_i \leq 150) \\ &= \int_{-\infty}^{150} f(x)\, dx \\ &= 100 \int_{100}^{150} \frac{dx}{x^2} \end{aligned}$$

Thus

$$\mathbf{P}(E_i) = \frac{1}{3}$$

Example: radio tube (3)

Model for the set of tubes: Define

$$Z_i = \mathbf{1}_{E_i}, \qquad Z = \sum_{i=1}^{5} Z_i$$

Then

$$Z \sim \text{Bin}\left(5, \frac{1}{3}\right)$$

and we look for

$$\mathbf{P}(Z = 2)$$

Conclusion:

$$\mathbf{P}(Z = 2) = \binom{5}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^3 \simeq 33\%$$

Outline

General definition

Definition 2.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X : S \to \mathbb{R}$ a continuous random variable
- f = density of X

Then the expected value of X is defined by

$$\mathbf{E}[X] = \int_{\mathbb{R}} x\, f(x)\, dx$$

Heuristics for the definition

Recall the discrete case:

$$\mathbf{E}[X] = \sum_{i \geq 1} x_i \, \mathbf{P}(X = x_i)$$

Continuous case analog: We have

$$f(x)\, dx \simeq \mathbf{P}(x \leq X \leq x + dx)$$

Thus

$$\begin{aligned} \mathbf{E}[X] &\simeq \sum x_i \, \mathbf{P}(x_i \leq X \leq x_i + dx) \\ &\simeq \int_{\mathbb{R}} x \, f(x)\, dx \end{aligned}$$

Simple example (1)

Density of X: Consider X with density

$$f(x) = 2x\,\mathbf{1}_{[0,1]}(x)$$

Simple example (2)

Recall: We consider X with density

$$f(x) = 2x\,\mathbf{1}_{[0,1]}(x)$$

Expected value:

$$\begin{aligned}\mathbf{E}[X] &= \int_{\mathbb{R}} x\, f(x)\, dx \\ &= \int_0^1 2x^2\, dx \\ &= \frac{2}{3}\end{aligned}$$

Expression for $\mathbf{E}[X]$ when $X \geq 0$

Proposition 3.

Let

- X continuous random variable
- f density of X

Hypothesis:

$$X \geq 0$$

Then

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > y)\, dy \tag{1}$$

Proof

Expression for the rhs:

$$\int_0^\infty \mathbf{P}(X > y)\,dy \;=\; \int_0^\infty \left(\int_y^\infty f(x)\,dx\right)\,dy$$

Apply Fubini: Invert the order of integration

$$\begin{aligned}\int_0^\infty \mathbf{P}(X > y)\,dy &= \int_0^\infty \left(\int_0^x dy\right)\,f(x)\,dx \\ &= \int_0^\infty x\,f(x)\,dx \\ &= \mathbf{E}[X]\end{aligned}$$

Definition of $\mathbf{E}[g(X)]$

Proposition 4.

Let

- X continuous random variable
- f density of X
- g real valued function

Then

$$\mathbf{E}\left[g(X)\right] = \int_{\mathbb{R}} g(x)\, f(x)\, dx \tag{2}$$

Simple example – Ctd (1)

Density of X: Consider X with density

$$f(x) = 2x\, \mathbf{1}_{[0,1]}(x)$$

Question: Compute

$$\mathbf{E}[X^3]$$

Simple example – Ctd (2)

Recall: We consider X with density

$$f(x) = 2x\, \mathbf{1}_{[0,1]}(x)$$

Expected value for $g(x) = x^3$:

$$\begin{aligned} \mathbf{E}[X^3] &= \int_{\mathbb{R}} x^3 f(x)\, dx \\ &= \int_0^1 2x^4\, dx \\ &= \frac{2}{5} \end{aligned}$$

Proof of Proposition 4

Hypothesis:

We assume $X \geq 0$ and $g(X) \geq 0$ for the proof

Expression with (1):

$$\begin{aligned} \mathbf{E}[g(X)] &= \int_0^\infty \mathbf{P}(g(X) > y)\, dy \\ &= \int_0^\infty \left(\int_{\{x;\, g(x) > y\}} f(x)\, dx \right) dy \end{aligned}$$

Apply Fubini: Invert the order of integration

$$\begin{aligned} \mathbf{E}[g(X)] &= \int_0^\infty \left(\int_0^{g(x)} dy \right) f(x)\, dx \\ &= \int_0^\infty g(x)\, f(x)\, dx \end{aligned}$$

Expectation and linear transformations

Proposition 5.

Let

- X continuous random variable
- f density of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{E}[aX + b] = a\,\mathbf{E}[X] + b$$

Proof

Application of relation (2):

$$\begin{aligned} \mathbf{E}[aX+b] &= \int_{\mathbb{R}} (a\,x+b)\; f(x)\,dx \\ &= a\int_{\mathbb{R}} x\,f(x)\,dx + b\int_{\mathbb{R}} f(x)\,dx \\ &= a\,\mathbf{E}[X]+b \end{aligned}$$

Definition of variance

Definition 6.

Let

- X continuous random variable
- f density of X
- $\mu = \mathbf{E}[X]$

Then we define $\mathbf{Var}(X)$ by

$$\mathbf{Var}(X) = \mathbf{E}\left[(X-\mu)^2\right]$$

Alternative expression for the variance

Proposition 7.

Let

- X continuous random variable
- f density of X
- $\mu = \mathbf{E}[X]$

Then $\mathbf{Var}(X)$ can be written as

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - \mu^2 = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Simple example – Ctd

Density of X: Consider X with density

$$f(x) = 2x\,\mathbf{1}_{[0,1]}(x)$$

Expected value for $g(x) = x^2$:

$$\mathbf{E}[X^2] = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^1 2x^3\,dx = \frac{1}{2}$$

Variance of X:

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}$$

Variance and linear transformations

Proposition 8.

Let

- X continuous random variable
- f density of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{Var}\,(aX + b) = a^2\,\mathbf{Var}(X)$$

Outline

Uniform random variable (1)

Notation:

$$X \sim \mathcal{U}([\alpha, \beta]), \text{ with } \alpha < \beta$$

State space:

$$[\alpha, \beta]$$

Density:

$$f(x) = \frac{1}{\beta - \alpha} \mathbf{1}_{[\alpha,\beta]}(x)$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{\alpha + \beta}{2}, \qquad \mathbf{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$

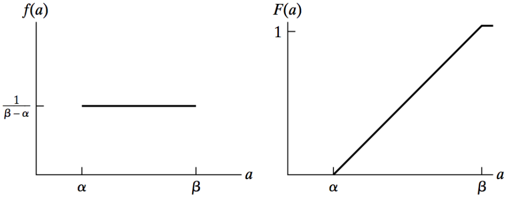
Uniform random variable (2)

Use:

- $\mathcal{U}([0,1])$ only r.v directly accessible on a computer
  $\hookrightarrow$ `rand` function

Example of computation: if $X \sim \mathcal{U}([8,10])$, then

$$\mathbf{P}(7.5 < X < 9.5) = \frac{1}{2}\int_8^{9.5} dx = \frac{9.5-8}{2} = \frac{3}{4}$$



Bertrand's paradox (1)

Experiment:

- Draw a random chord of a circle with center O and radius r

Question: Compute

$$p = \text{probability that the chord is larger than the side of the inscribed equilateral triangle}$$

Bertrand's paradox (2)

Model 1:

- Chord determined by its distance D to the center
- $D \sim \mathcal{U}([0, r])$

Computation of p under Model 1:

$$p = \mathbf{P}\left(D \leq \frac{r}{2}\right) = \frac{1}{2}$$

Bertrand's paradox (3)

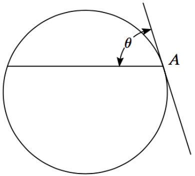
Model 2:

- Chord parametrized by θ
- $\theta =$ angle between chord and tangent
- $\theta \sim \mathcal{U}([0, 90])$

Computation of p under Model 2:

According to tangent-chord theorem

$$p = \mathbf{P}\left(60 < \theta < 90\right) = \frac{90 - 60}{90} = \frac{1}{3}$$



Outline

Normal random variable (1)

Notation:

$$\mathcal{N}(\mu, \sigma^2), \text{ with } \mu \in \mathbb{R} \text{ and } \sigma^2 > 0$$

State space:

$$\mathbb{R}$$

Density:

$$f(x) = \frac{1}{\sqrt{2\pi\,\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Expected value and variance:

$$\mathbf{E}[X] = \mu, \qquad \mathbf{Var}(X) = \sigma^2$$

Normal random variable (2)

Use:
Quantities which depend on a large number of small parameters

Numerous examples in:

- Biology
- Physics and industry
- Economics

Normal random variable (3)

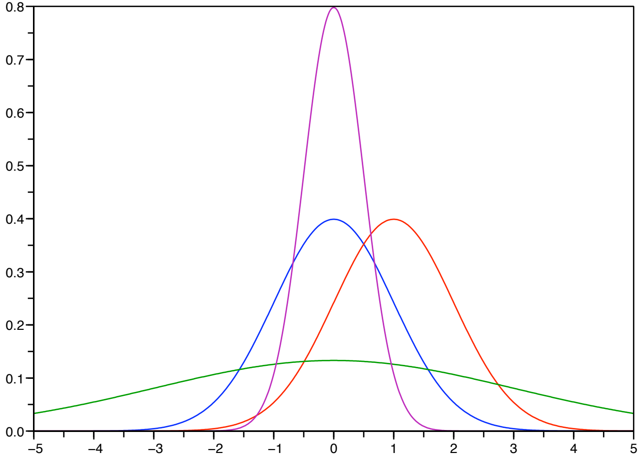


Figure: densities for $\mathcal{N}(0, 1)$, $\mathcal{N}(1, 1)$, $\mathcal{N}(0, 9)$, $\mathcal{N}(0, 1/4)$.

Normal random variable (4)

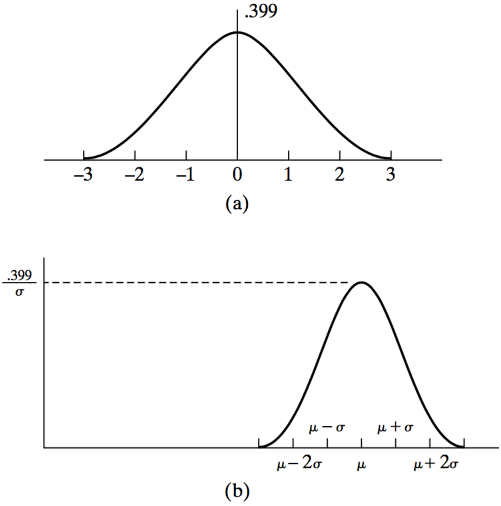


Figure: Densities for (a) $\mathcal{N}(0,1)$ (b) $\mathcal{N}(\mu,\sigma^2)$

Normal r.v and linear transformations

Proposition 9.

Let

- $X \sim \mathcal{N}(0,1)$
- $\mu \in \mathbb{R}$ and $\sigma > 0$
- Set $Y = \sigma X + \mu$

Then

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

Cdf for a normal r.v

Function Φ: For $X \sim \mathcal{N}(0,1)$ and $x \geq 0$, set

$$\Phi(x) = \mathbf{P}(X \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \, dy$$

Problem with Φ:

- No algebraic expression
- Numerical approximation needed
- Use of tables

Property of Φ: For $x \geq 0$,

$$\Phi(-x) = 1 - \Phi(x)$$

Table for Φ

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936

Simple normal computation (1)

Definition of a random variable: We let

$$X \sim \mathcal{N}\left(\mu = 3,\, \sigma^2 = 9\right)$$

Questions: Compute

1. $\mathbf{P}(2 < X < 5)$
2. $\mathbf{P}(X > 0)$
3. $\mathbf{P}(|X - 3| > 6)$

Simple normal computation (2)

Change of variable: We define $Z \sim \mathcal{N}(0,1)$ by

$$Z = \frac{X-\mu}{\sigma} = \frac{X-3}{3}$$

First question: We have

$$\begin{aligned} \mathbf{P}(2 < X < 5) &= \mathbf{P}\left(-\frac{1}{3} < Z < \frac{2}{3}\right) \\ &= \Phi\left(\frac{2}{3}\right) - \left(1 - \Phi\left(\frac{1}{3}\right)\right) \\ &\simeq .3779 \end{aligned}$$

Simple normal computation (2)

Second question: We have

$$\begin{aligned} \mathbf{P}(X > 0) &= \mathbf{P}(Z > -1) \\ &= 1 - \Phi(-1) \\ &= \Phi(1) \\ &\simeq .8413 \end{aligned}$$

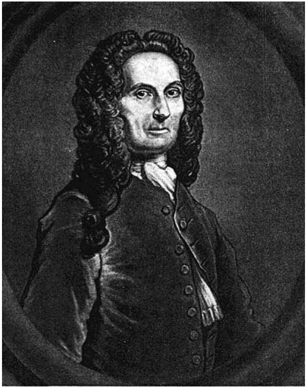
Third question: We have

$$\begin{aligned} \mathbf{P}(|X - 3| > 6) &= \mathbf{P}(|Z| > 2) \\ &= 1 - \Phi(2) + \Phi(-2) \\ &= 2[1 - \Phi(2)] \\ &\simeq .0456 \end{aligned}$$

Abraham de Moivre

Some facts about de Moivre:

- Lifespan: 1667-1754, in $\simeq$ Paris, London
- Ousted from France as a protestant
  $\hookrightarrow$ in $\simeq$ 1687
- In London lived from
  - Private lessons
  - Assisting gamblers in a coffee house
- Contributions in math
  - Stirling's formula
  - First central limit theorem
  - First results on Poisson distribution



DeMoivre-Laplace theorem

Theorem 10.

Let

- $n \geq 1$, $p \in (0,1)$
- $X_n \sim \text{Bin}(n,p)$
- $a < b$

Then

$$\lim_{n\to\infty} \mathbf{P}\left(a < \frac{X_n - np}{(np(1-p))^{1/2}} < b\right) = \Phi(b) - \Phi(a)$$

Empirical rule:

Accept approximation as long as $np(1-p) \geq 10$

Binomial converging to normal: illustration

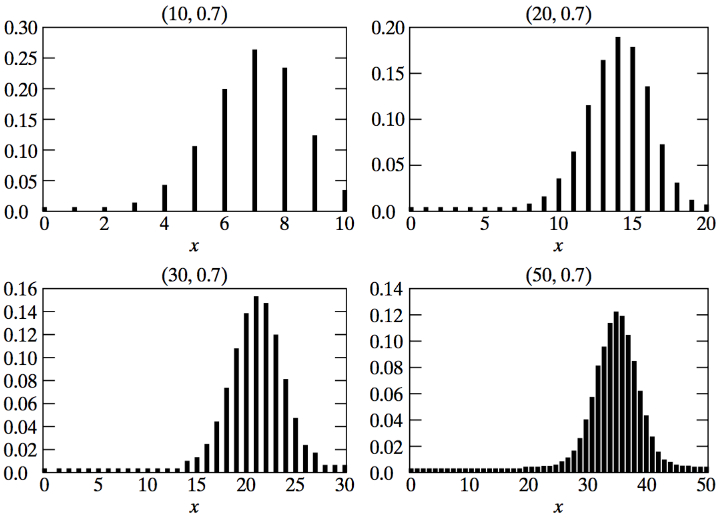


Figure: Binomial histograms for different values of (n, p)

Example: enrollment overbooking (1)

Situation:

- Ideal size of a first-year class at a particular college is 150 students.
- Data: on average, only 30% of those accepted for admission will actually attend
- College policy: approve the applications of 450 students.

Question:

Compute the probability that
more than 150 first-year students attend this college.

Example: enrollment overbooking (2)

Notation: We define

- $n = 450$, $p = .3$
- $X_i = \mathbf{1}_{(i\text{-th accepted student attends})}$, for $i = 1, \ldots, n$

Hypothesis:

- X_i i.i.d with common law $\mathcal{B}(p)$

Random variable of interest: Set

$$X = \# \text{ students that will attend}$$

Then

$$X = \sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$$

Example: enrollment overbooking (3)

Normal approximation: We look for

$$\mathbf{P}\left(X \geq 150.5\right) = \mathbf{P}\left(\frac{X - 450 \times 0.3}{(450 \times 0.3 \times 0.7)^{1/2}} \geq \frac{150.5 - 450 \times 0.3}{(450 \times 0.3 \times 0.7)^{1/2}}\right)$$

Therefore by DeMoivre-Laplace,

$$\mathbf{P}\left(X \geq 150.5\right) \simeq 1 - \Phi(1.59) \simeq 5.59\%$$

Outline

Exponential random variable (1)

Notation:

$$\mathcal{E}(\lambda), \text{ with } \lambda > 0$$

State space:

$$\mathbb{R}_+ = [0, \infty)$$

Density:

$$f(x) = \lambda e^{-\lambda x} \mathbf{1}_{\mathbb{R}_+}(x)$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{1}{\lambda}, \qquad \mathbf{Var}(X) = \frac{1}{\lambda^2}$$

Exponential random variable (2)

Use: Waiting time between

- 2 customer arrivals in a shop on a typical afternoon
- Bus arrivals at a bus stop
- Two jobs on a server from 12am to 6am

Empirical rule:

Number of arrivals given by a Poisson random variable

$$\Longrightarrow$$

Inter arrivals given by exponential random variables

Tail probability: If $X \sim \mathcal{E}(\lambda)$, then for $x \geq 0$ we have

$$\mathbf{P}(X > x) = \int_x^\infty \lambda \, e^{-\lambda z} \, dz = e^{-\lambda x}$$

Graphing an exponential law

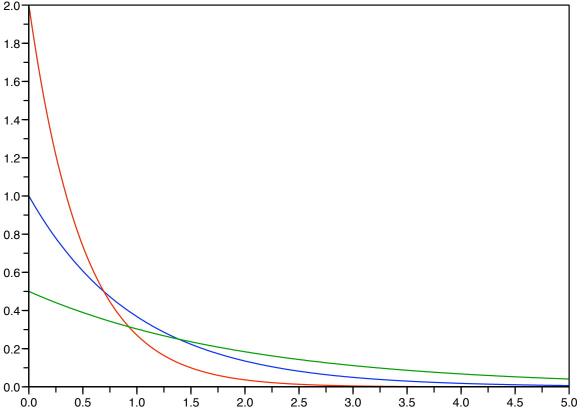


Figure: $\mathcal{E}(1)$, $\mathcal{E}(2)$, $\mathcal{E}(1/2)$. x-axis: x. y-axis: $f(x)$

Memoryless property

Proposition 11.

Let

- X be continuous random variable

Then X satisfies the memoryless property

$$\mathbf{P}(X > s + t | X > t) = \mathbf{P}(X > s)$$

if and only if there exists $\lambda > 0$ such that

$$X \sim \mathcal{E}(\lambda)$$

Proof of $\Longrightarrow$ (1)

Functional equation: Set

$$\bar{F}(x) = \mathbf{P}(X > x)$$

Then if X is memoryless, $\bar{F}$ satisfies

$$g(s+t) = g(s)\, g(t) \tag{3}$$

Value of g on rationals: If g satisfies (3), then

$$g\left(\frac{1}{n}\right) = (g(1))^{1/n}, \qquad g\left(\frac{m}{n}\right) = (g(1))^{m/n}$$

Proof of $\Longrightarrow$ (2)

Expression for $g(1)$:
We have $g(1) = [g(1/2)]^2 \geq 0$. Thus there exists $\lambda \in \mathbb{R}$ such that

$$g(1) = e^{-\lambda}$$

Value of g on rationals (2): We have found that for $x \in \mathbb{Q}_+$,

$$g(x) = e^{-\lambda x}$$

Conclusion: By continuity of g, for all $x \in \mathbb{R}_+$ we have

$$g(x) = e^{-\lambda x}$$

Example: car battery (1)

Situation:

- Number of miles that a car can run before its battery wears out is exponentially distributed
- Average value of 10k miles
- We have already run 3k miles with the battery
- We wish to take a 5k trip

Question: Probability to complete the trip without having to replace the car battery?

Example: car battery (1)

Model:

- $X = \#$ miles before battery wears out
- $X \sim \mathcal{E}(\lambda)$
- $\lambda = \frac{1}{\mathbf{E}[X]} = \frac{1}{10}$
- We wish to compute $\mathbf{P}(X > 3 + 5 | X > 3)$

Computation:

$$\mathbf{P}(X > 3 + 5 | X > 3) = \mathbf{P}(X > 5) = e^{-\frac{1}{2}} \simeq 0.604$$

Hazard rate function (1)

Definition 12.

Let

- X positive continuous random variable
- Density f, cdf F
- $\bar{F} = 1 - F$

Then the hazard rate function is given by

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}$$

Hazard rate function (2)

Interpretation: λ is a failure rate, i.e

$$\mathbf{P}\left(X \in [t, t+dt) \middle| X > t\right) \simeq \lambda(t)\, dt$$

Exponential case: If $X \sim \mathcal{E}(\lambda)$, we have

$$\lambda(t) = \lambda$$

Hazard rate function (3)

Cdf from λ: from the relation

$$\lambda(t) = \frac{F'(t)}{1 - F(t)},$$

we get

$$F(t) = 1 - \exp\left(-\int_0^t \lambda(s)\, ds\right)$$

Survival probability from λ: For $a, b \geq 0$,

$$\mathbf{P}\left(X > a + b \middle| X > a\right) = \exp\left(-\int_a^{a+b} \lambda(s)\, ds\right)$$

Example: smokers survival (1)

Data:

- Death rate of smokers = twice death rate of non smokers
- Consider 2 40-years old persons, 1 S and 1 N
- We wish to compare their probability to survive until 50

Model: Let

$$\lambda_n = \text{ hazard rate for N}, \qquad \lambda_s = \text{ hazard rate for S}$$

Then

$$\lambda_s = 2\,\lambda_n$$

Example: smokers survival (2)

Compute:

$$\begin{aligned}
\mathbf{P}(S > 50 | S > 40) &= \exp\left(-\int_{40}^{50} \lambda_s(r)\, dr\right) \\
&= \exp\left(-2\int_{40}^{50} \lambda_n(r)\, dr\right) \\
&= \left[\mathbf{P}(N > 50 | N > 40)\right]^2
\end{aligned}$$

Outline

A quote by Von Neumann

Quote:

- In mathematics,
  you don't understand things.
  You just get used to them.

Gamma random variable (1)

Notation:

$$\Gamma(\alpha, \lambda), \text{ with } \alpha, \lambda > 0$$

State space:

$$\mathbb{R}_+ = [0, \infty)$$

Density:

$$f(x) = \frac{\lambda\, e^{-\lambda x}\, (\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\, \mathbf{1}_{\mathbb{R}_+}(x), \quad \text{where} \quad \Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1}\, dy$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{\alpha}{\lambda}, \qquad \mathbf{Var}(X) = \frac{\alpha}{\lambda^2}$$

Gamma random variable (2)

Use 1: Assume

- $\{T_i;\ i \geq 1\}$ i.i.d with common law $\mathcal{E}(\lambda)$
- $T = \sum_{i=1}^{n} T_i$

Then $T \sim \Gamma(n, \lambda)$

Use 2: Assume

- $\{X_i;\ i \geq 1\}$ i.i.d with common law $\mathcal{N}(0, 1)$
- $Z = \sum_{i=1}^{n} X_i^2$

Then

- $Z \sim \Gamma(\frac{n}{2}, \frac{1}{2})$
- Z is called a chi-squared r.v with n degrees of freedom

Weibull random variable (1)

Notation:

$$\mathcal{W}(\alpha, \beta, \nu), \text{ with } \alpha, \beta > 0 \text{ and } \nu \in \mathbb{R}$$

State space:

$$(\nu, \infty)$$

Cdf:

$$F(x) = \left[1 - \exp\left(-\left(\frac{x-\nu}{\alpha}\right)^{\beta}\right)\right] \mathbf{1}_{(\nu,\infty)}(x),$$

Expected value and variance:

$$\mathbf{E}[X] = \nu + \alpha\,\Gamma\left(1 + \frac{1}{\beta}\right), \quad \mathbf{Var}(X) = \alpha^2 \left[\Gamma\left(1 + \frac{2}{\beta}\right) - \left(\Gamma\left(1 + \frac{1}{\beta}\right)\right)^2\right]$$

Weibull random variable (2)

Use:

- Widely used for lifetimes in engineering systems
- Versatile in order to model aging

Hazard rate function:

$$\lambda(x) = \frac{\beta}{\alpha}\left(\frac{x-\nu}{\alpha}\right)^{\beta-1}$$

Cauchy random variable (1)

Notation:

$$\text{Cauchy}(\alpha), \text{ with } \alpha \in \mathbb{R}$$

State space:

$$\mathbb{R}$$

Density:

$$f(x) = \frac{1}{\pi} \frac{1}{1 + (x - \alpha)^2}$$

Expected value and variance:

Not defined (divergent integrals)!

Cauchy random variable (2)

Use 1: Trigonometric function of a uniform r.v
Namely if

- $X \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$
- $Y = \tan(X)$

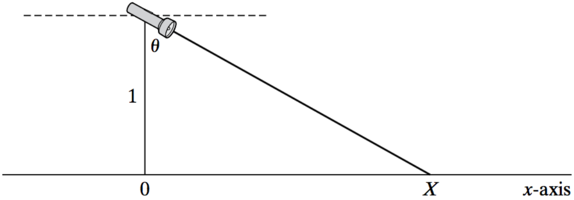
Then $Y \sim \text{Cauchy} \equiv \text{Cauchy}(0)$

Use 2:
Typical example of r.v with no mean

Example: beam (1)

Experiment:

- Narrow-beam flashlight spun around its center
- Center located a unit distance from the x-axis
- $X =$ point at which the beam intersects the x-axis
  when the flashlight has stopped spinning



Example: beam (2)

Model:

- We assume $\theta \sim \mathcal{U}([-\frac{\pi}{2}, \frac{\pi}{2}])$
- We have $X \sim \tan(\theta)$

Conclusion:

$$X \sim \text{Cauchy}$$

Beta random variable (1)

Notation:

$$\text{Beta}(a, b), \text{ with } a, b > 0$$

State space:

$$[0, 1]$$

Density:

$$f(x) = \frac{1}{B(a,b)}\, x^{a-1}(1-x)^{b-1}\, \mathbf{1}_{[0,1]}(x)$$

Expected value and variance:

$$\mathbf{E}[X] = \frac{a}{a+b}, \qquad \mathbf{Var}(X) = \frac{a\,b}{(a+b)^2(a+b+1)}$$

Beta random variable (2)

Beta function: In the definition of f we have set

$$B(a,b) = \int_0^1 s^{a-1}(1-s)^{b-1}\,ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

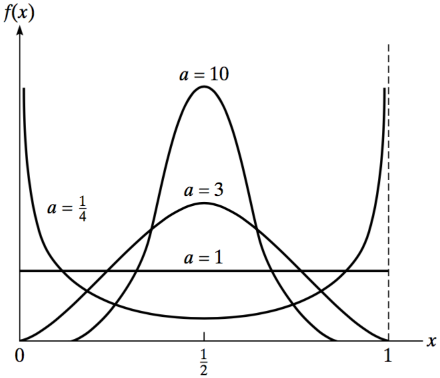
Use:
Models for which we know that $X \in [c,d]$

Behavior of f:

- If $a = b$, then f symmetric with respect to $\frac{1}{2}$
  $\hookrightarrow$ as $a \nearrow \infty$, more weight given to $\frac{1}{2}$
- If $b > a$, f is skewed to the left
- If $a > b$, f is skewed to the right

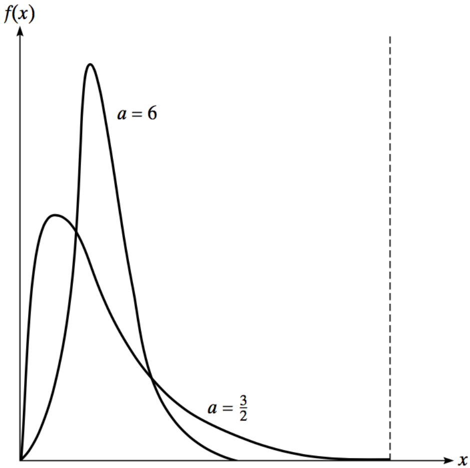
Beta random variable (3)

Examples of f with $a = b$:



Beta random variable (4)

Examples of f with $b = 19a$: This also means $\mathbf{E}[X] = \frac{1}{20}$



Outline

Characterizing r.v by expected values

Notation:

$C_b(\mathbb{R}) \equiv$ set of continuous and bounded functions on $\mathbb{R}$.

Theorem 13.

Let X be a r.v. We assume that

$$\mathbf{E}[\varphi(X)] = \int_{\mathbb{R}} \varphi(x)\, f(x)\, dx, \quad \text{for all functions } \varphi \in C_b(\mathbb{R}).$$

Then X is continuous, with density f.

Application: change of variable

Problem: Let

- X random variable with density f.
- Set $Y = h(X)$ with $h : \mathbb{R} \to \mathbb{R}$.

We wish to find the density of Y.

Application: change of variable (2)

Recipe: One proceeds as follows

1. For $\varphi \in C_b(\mathbb{R})$, write

$$\mathbf{E}[\varphi(Y)] = \mathbf{E}[\varphi(h(X))] = \int_{\mathbb{R}} \varphi(h(x))\, f(x)\, dx.$$

2. Change variables $y = h(x)$ in the integral.
   After some elementary computations we get

$$\mathbf{E}[\varphi(Y)] = \int_{\mathbb{R}} \varphi(y)\, g(y)\, dy.$$

3. This characterizes Y, which admits a density g

Example: normal r.v and linear transformations

Proposition 14.

Let

- $X \sim \mathcal{N}(0,1)$
- $\mu \in \mathbb{R}$ and $\sigma > 0$
- Set $Y = \sigma X + \mu$

Then

$$Y \sim \mathcal{N}(\mu, \sigma^2)$$

Proof

Recipe, item 1: for $\varphi \in C_b(\mathbb{R})$, write

$$\mathbf{E}[\varphi(Y)] = \mathbf{E}[\varphi(\sigma X + \mu)] = \int_{\mathbb{R}} \varphi(\sigma x + \mu) \frac{e^{-x^2/2}}{\sqrt{2\pi}} \, dx.$$

Recipe, item 2: Change of variable: $y = \sigma x + \mu$:

$$\mathbf{E}[\varphi(Y)] = \int_{\mathbb{R}} \varphi(y) \, g(y) \, dx, \quad \text{with} \quad g(y) = \frac{e^{-(y-\mu)^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2}}.$$

Recipe, item 3:

Y is continuous with density g, therefore $Y \sim \mathcal{N}(\mu, \sigma^2)$.

Example: waiting time

Question 1: At Dr Gesund's office, the waiting time (in mn) is modeled by a r.v $Y = 5 + X$, where $X \sim \mathcal{E}(\lambda)$ with $\lambda = 1/2$. Find the density of Y.

We find $f_Y(y) = \lambda e^{-\lambda(y-5)} \mathbf{1}_{[5,\infty)}(y)$.

Question 2: The typical patient dissatisfaction is measured by the r.v $Z = \ln(X)$. Find the density of Z.

We find $f_Z(z) = \lambda \exp(-\lambda e^z + z)$.

Change of variable: general result

Theorem 15.

Let

- X continuous random variable
- Density: f_X
- g strictly monotonic differentiable function
- $Y = g(X)$

Then Y has a density f_Y given by

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\left|\left(g^{-1}(y)\right)'\right| \mathbf{1}_{\{y=g(x) \text{ for some } x\}}$$