$E[X]$ for $X \sim U([\alpha,\beta])$

$$E[X] = \int_{\mathbb{R}} x\, f(x)\, dx$$

$$= \int_{\mathbb{R}} x \, \frac{1}{\beta-\alpha} \, \mathbb{1}_{[\alpha,\beta]}(x)\, dx$$

$$= \left(\int_{\alpha}^{\beta} x \, dx \right) \times \frac{1}{\beta-\alpha}$$

$$= \frac{x^2}{2} \Big|_{\alpha}^{\beta} \times \frac{1}{\beta-\alpha}$$

$$= \frac{1}{2(\beta-\alpha)} \underbrace{\left(\beta^2-\alpha^2 \right)}_{(\beta-\alpha)(\beta+\alpha)}$$

$$= \frac{\cancel{(\beta-\alpha)}(\beta+\alpha)}{2\cancel{(\beta-\alpha)}}$$

$$= \frac{\beta+\alpha}{2}$$

$E[X^2]$ for $X \sim U([\alpha, \beta])$

$$E[X^2] = \int_{\mathbb{R}} x^2 \overbrace{f(x)}^{= \frac{1}{\beta\alpha} \mathbb{1}_{[\alpha,\beta]}(x)} dx$$

$$= \frac{1}{\beta-\alpha} \int_\alpha^\beta x^2 \, dx$$

$$= \frac{1}{3(\beta-\alpha)} \, x^3 \Big|_\alpha^\beta$$

$$= \frac{\overbrace{\beta^3 - \alpha^3}^{= \cancel{(\beta-\alpha)}(\beta^2 + \beta\alpha + \alpha^2)}}{3\cancel{(\beta-\alpha)}}$$

$$= \frac{1}{3}\left(\beta^2 + \beta\alpha + \alpha^2\right)$$

$Var(X)$ for $X \sim U([\alpha, \beta])$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{1}{3}(\beta^2 + \beta\alpha + \alpha^2) - \left(\frac{\beta+\alpha}{2}\right)^2$$

$$= \frac{1}{3}(\beta^2 + \beta\alpha + \alpha^2) - \frac{1}{4}(\beta+\alpha)^2$$

$$= \frac{1}{12}\left\{4(\beta^2 + \beta\alpha + \alpha^2) - 3(\beta^2 + 2\alpha\beta + \alpha^2)\right\}$$

Check

$$= \frac{1}{12}(\beta-\alpha)^2$$

Rmk If $X \sim U([\alpha, \beta])$,

$$\sigma_X = \sqrt{Var(X)}$$

$$= \frac{(\beta-\alpha)}{2\sqrt{3}}$$

$\longrightarrow$ proportional to the length of $[\alpha, \beta]$

Computation for $\mathcal{U}([\overset{\alpha}{8},\overset{\beta}{10}])$

$$P(7.5 < X < 9.5)$$

$$= \int_{7.5}^{9.5} \overbrace{f(x)}^{\frac{1}{2}\mathbb{1}_{[8,10]}(x)} \, dx$$

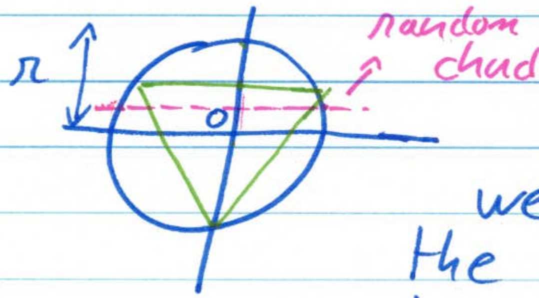
$$= \frac{1}{2} \int_{7.5}^{9.5} \mathbb{1}_{[8,10]}(x) \, dx$$

$$= \frac{1}{2} \int_{8}^{9.5} dx$$

$$= \frac{1}{2} (9.5 - 8)$$

$$= \frac{3}{4}$$

Bertrand's paradox (1)



Model for a random chord: we assume that the distance of the chord to O is

$$D \sim \mathcal{U}([0,r])$$

Rmk: If we have an inscribed equilateral triangle, then $D = \frac{r}{2}$

Then

$p = P($random chord larger than side of equilateral triangle$)$

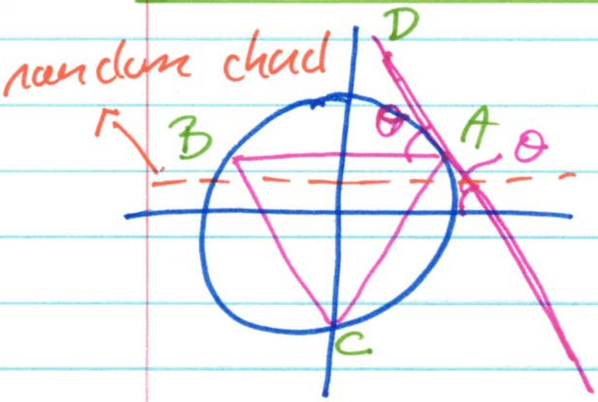
$$= P\left(D \le \frac{r}{2}\right)$$

$$= \int_0^{r/2} \frac{1}{r} \mathbb{1}_{[0,r]}(x)\,dx$$

$$= \frac{1}{r} \times \frac{r}{2} \quad = \frac{1}{2}$$

Thus $\quad p = \frac{1}{2}$

Bertrand's paradox (2)



Model 2: Set

θ = angle(chord, tangent)

Then $\theta \sim \mathcal{U}([0, 90])$

In the equilateral situation

$$\angle BAD = \angle BCA = 60 = \theta$$

In this case

$p = P($random chord longer than side of equilateral triangle$)$

$$= P(60 < \theta < 90)$$

$$= \frac{1}{90} \int_{60}^{90} dx$$

$$= \frac{90-60}{90}$$

$$= \frac{1}{3}$$

Thus $p = \frac{1}{3}$

(paradox)