Normal r.v, $E[X]$ ; $\mu=0$ , $\sigma^2=1$

Then $f(x)=\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}$

We have

$$E[X]=\int_{\mathbb{R}} x\, f(x)\, dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \overbrace{x\, e^{-\frac{x^2}{2}}}^{\equiv g(x)}\, dx$$

Then

$$g(-x)=-x\, e^{-\frac{(-x)^2}{2}}$$

$$=-x\, e^{-\frac{x^2}{2}} = -g(x)$$

Thus

$$\int_{-\infty}^{\infty} g(x)\, dx = 0$$

Conclusion: if $\mu=0$ , then $E[X]=0$

⚠ Before writing $\int_{-\infty}^{\infty} g(x)\,dx = 0$, we should check that the integral is convergent (see calc course)

Basic summary about convergence of $\int$

(i) $\int_0^1 \frac{dx}{x^r} < \infty$ iff $r < 1$

convergent

(ii) $\int_1^\infty \frac{dx}{x^r} < \infty$ iff $r > 1$

(iii) If $g$ is such that

$\lim_{x \to \infty} x^r g(x) = 0$, then

$\int_\infty g(x)\,dx < \infty$

Application: check that $\int_{-\infty}^{\infty} g(x)dx$ well defined if $g(x) = x e^{-\frac{x^2}{2}}$.

We have, for $r = 2$ $(>1)$

$$\lim_{x \to \pm\infty} x^2 g(x)$$

polynomial  exp. function

$$= \lim_{x \to \pm\infty} x^3 e^{-\frac{x^2}{2}}$$

$$= 0$$

Therefore, according (iii), we have $\int_{\pm\infty} g(x)\,dx < \infty$ and $E[X]$ well-defined

Var(X) if $X \sim \mathcal{N}(\overset{\mu}{0}, \overset{\sigma^2}{1})$

$$Var(X) = E[X^2] - (\overset{0}{E[X]})^2$$

$$= E[X^2]$$

$$E[X^2] = \int_{\mathbb{R}} x^2 \overset{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{f(x)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 \, e^{-\frac{x^2}{2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \underbrace{x}_{u} \, \underbrace{\overset{h'}{x} \, e^{-\overset{h}{\frac{x^2}{2}}}}_{v'} \, dx$$

i.b.p

$$= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{-x\, e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} 1 \times e^{-\frac{x^2}{2}} dx \right\}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx$$

f density function

$$= \int_{-\infty}^{\infty} f(x)\, dx = 1$$

Conclusion: If $\mu = 0$, $\sigma^2 = 1$,

$Var(X) = 1$