

①

Normal r.v.,  $E[X]$  ;  $\mu=0$  ,  $\sigma^2=1$

$$\text{Then } f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

We have

$$E[X] = \int_{\mathbb{R}} x f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \underbrace{e^{-\frac{x^2}{2}}}_{= g(x)} dx$$

Then

$$\underline{g(-x)} = -x e^{-\frac{(-x)^2}{2}}$$

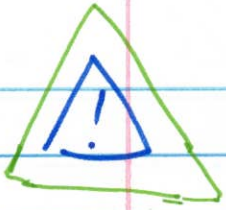
$$= -x e^{-\frac{x^2}{2}} = -\underline{g(x)}$$

Thus

$$\int_{-\infty}^{\infty} g(x) dx = 0$$

Conclusion: if  $\mu=0$  , then  
 $E[X] = 0$

②



Before writing  $\int_{-\infty}^{\infty} g(x) dx = 0$ ,  
we should check that the integral  
is convergent (see calc course)

Basic summary about convergence of  $\int$

(i)  $\int_0^1 \frac{dx}{x^r} < \infty$   <sup>$r >$  convergent</sup> iff  $r < 1$

(ii)  $\int_1^{\infty} \frac{dx}{x^r} < \infty$  iff  $r > 1$

(iii) If  $g$  is such that

$$\lim_{x \rightarrow \infty} x^r g(x) = 0, \text{ then}$$

$$\int_{\infty} g(x) dx < \infty$$

(3)

Application: check that  $\int_{-\infty}^{\infty} g(x) dx$   
well defined if  $g(x) = x e^{-\frac{x^2}{2}}$ .

We have, for  $r = 2 (> 1)$

$$\lim_{x \rightarrow \pm\infty} x^2 |g(x)|$$

*polynomial*  $\rightarrow$  *exp. function*

$$= \lim_{x \rightarrow \pm\infty} x^3 e^{-\frac{x^2}{2}}$$

$$= 0$$

Therefore, according (iii), we

have  $\int_{-\infty}^{\infty} |g(x)| dx < \infty$

and  $E[X]$  well-defined

(4)

Var(x) if  $X \sim N(0,1)$

$$\text{Var}(x) = E[X^2] - (E[X])^2$$

$$= E[X^2]$$

$$E[X^2] = \int_{\mathbb{R}} x^2 f(x) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^2 e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \underbrace{x}_{u'} \underbrace{e^{-\frac{x^2}{2}}}_{u} dx$$

$$\stackrel{i.b.p}{=} \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{-x e^{-\frac{x^2}{2}}}_{=0} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 1 \times e^{-\frac{x^2}{2}} dx \right\}$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

$$= \int_{-\infty}^{\infty} f(x) dx = 1 \quad \leftarrow \text{density function}$$

Conclusion : If  $\mu=0, \sigma^2=1,$   
 $\text{Var}(x) = 1$