$E[X]$, $X \sim \mathcal{E}(\lambda)$

$$E[X] = \int_{\mathbb{R}} x f(x)\, dx$$

$$= \int_{\mathbb{R}} x \ \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}_+}(x)\, dx$$

$$= \int_0^\infty \underbrace{x}_{u} \ \underbrace{\lambda e^{-\lambda x}}_{v'} \, dx$$

polynomial   damped exp

$$= x\left(-e^{-\lambda x}\right)\Big|_0^\infty$$

$$+ \int_0^\infty 1 \times e^{-\lambda x}\, dx$$

$$= 0 + \int_0^\infty e^{-\lambda x}\, dx$$

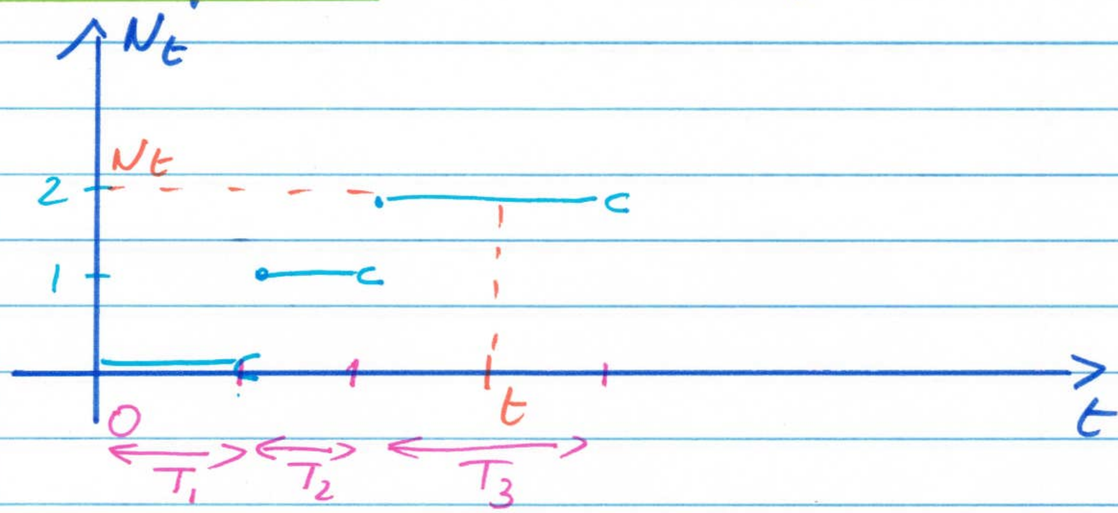
$$= -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^\infty$$

$$= -0 + \frac{1}{\lambda}$$

$$= \frac{1}{\lambda}$$

Rmk In order to compute $E[X^2]$, i.b.parts twice

Poisson process (random function)



Hyp: T_i's are $\perp\!\!\!\perp$, $T_i \sim \mathcal{E}(\lambda)$

Then one can prove that for a fixed $t \geq 0$,

$$N_t \sim \mathcal{P}(\lambda t)$$

Tail of an exp. r.v.

$X \sim \mathcal{E}(\lambda)$. Then for $x \geq 0$

$$1 - F(x) = P(X > x)$$

$$= \int_x^\infty f(z)\, dz$$

$$= \int_x^\infty \lambda e^{-\lambda z}\, dz$$

$$= -e^{-\lambda z}\Big|_x^\infty$$

$$= e^{-\lambda x}$$

If X cont. r.v.

$$\text{Memoryless} \quad \overset{\text{easy}}{\Longleftarrow} \quad \underset{\text{more involved}}{\Longrightarrow} \quad \mathcal{E}(\lambda)$$

Proof $X \sim \mathcal{E}(\lambda) \Rightarrow X$ memoryless

Let $X \sim \mathcal{E}(\lambda)$. Then

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$P(X > s+t \mid X > t)$$

$$= \frac{P((X > s+t) \cap (X > t))}{P(X > t)}$$

$$= \frac{P(X > s+t)}{P(X > t)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s}$$

$$= P(X > s)$$

Memoryless: modeling non aging system

If $X \sim \mathcal{E}(\lambda)$, $X \equiv$ lifetime of a system

then

prob to live s instants after time 0

$$\underbrace{P(X > t+s \mid X > t)}_{\text{prob. to live s instants after t given that we are still alive at time t}} = P(X > s)$$

Rmk: More realistic model

↳ Weibull r.v.

Cdf from λ:

$$\lambda(t) \overset{def}{=} \frac{f(t)}{\bar{F}(t)} = \frac{F'(t)}{1-F(t)}$$

$$\frac{F'(t)}{1-F(t)} = \lambda(t)$$

↳ Separable diff eq.

Integrating we get

$$F(t) = 1 - e^{-\int_0^t \lambda(s)\,ds}$$

$$\bar{F}(t) = 1 - F(t) = e^{-\int_0^t \lambda(s)\,ds}$$

Car battery example

X = # miles the car can run (k miles)

Hyp : $X \sim \mathcal{E}(\lambda)$, and

$$\lambda = \frac{1}{E[X]} = \frac{1}{10}$$

We wish to compute

$$P(X > 3+5 \mid X > 3)$$

$$\overset{\text{Memoryless}}{=} P(X > 5)$$

$$\overset{\text{Tail}}{=} e^{-5\lambda}$$

$$= e^{-5/10}$$

$$= e^{-1/2}$$

$$\simeq .604$$

Survival-example

$S =$ lifetime of the smoker

We wish to compute (for the smoker)

$$P(\cancel{X}S > 50 \mid S > 40)$$

survival func $= e^{-\int_{40}^{50} \lambda_s(x)\,dx}$ $\quad \lambda_s(x) \to 2\lambda_n(x)$

$$= e^{-2\int_{40}^{50} \lambda_n(x)\,dx}$$

$$= \left(e^{-\int_{40}^{50} \lambda_n(x)\,dx}\right)^2$$

$N =$ lifetime for non smoker

$$= \left(P(N > 50 \mid N > 40)\right)^2$$

Conclusion: If death rate is $\times 2$, then survival prob. is squared