Rmk on Thm 13

We have seen:

X r.v. with density f

$\Leftarrow$ $\longrightarrow$ contents of Thm 13

$\Longrightarrow$

$$E[\varphi(X)] = \int_{\mathbb{R}} \varphi(x)\, f(x)\, dx$$

for all $\varphi \in C_b(\mathbb{R})$

Proof of Prop 14

$\longrightarrow f_X(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ ②

Let $X \sim \mathcal{N}(0,1)$, $Y = \sigma X + \mu$

① Let $\varphi \in C_b(\mathbb{R})$. Then

$$E[\varphi(Y)] = E[\varphi(\sigma X + \mu)]$$

$$= \int_{-\infty}^{\infty} \varphi(\underbrace{\sigma x + \mu}_{y}) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

② CV : $\sigma x + \mu = y$

Thus $x = \frac{y-\mu}{\sigma}$ $\quad dx = \frac{dy}{\sigma}$

If $-\infty < x < \infty$, then $-\infty < y < \infty$

Thus

$$E[\varphi(Y)] = \int_{-\infty}^{\infty} \varphi(y) \frac{e^{-\frac{\left(\frac{y-\mu}{\sigma}\right)^2}{2}}}{\sqrt{2\pi}} \frac{dy}{\sigma}$$

$\sigma = \sqrt{\sigma^2}$

$$= \int_{-\infty}^{\infty} \varphi(y) \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dy$$

$\longrightarrow = g(y)$

③ $g(y)$ is the density of Y

$\Longrightarrow Y \sim \mathcal{N}(\mu, \sigma^2)$

Rmk on other methods for density of $h(X) = Y$.

1. In Ross' book
   ↳ based on cdf

2. Advantages of "my" method
   - no need to differentiate F_Y
   - same method works for vector-valued r.v. (see next chapter)

Docta's office example

$Y = 5 + X \quad , \quad X \sim \mathcal{E}(\lambda)$

Let $\varphi \in C_b(\mathbb{R})$. Then

1. $E[\varphi(Y)] = E[\varphi(5+X)]$

$= \int_{\mathbb{R}} \varphi(5+x)\, \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}_+}(x)\, dx$

$= \int_0^\infty \varphi(5+x)\, \lambda e^{-\lambda x}\, dx$

2. CV: $y = 5 + x$

Then $x = y - 5 \qquad dx = dy$

Bounds: $0 < x < \infty$ thus $5 < y < \infty$

$E[\varphi(Y)] = \int_5^\infty \varphi(y)\, \lambda e^{-\lambda(y-5)}\, dy$

⚠

$= \int_{\mathbb{R}} \varphi(y)\, \lambda e^{-\lambda(y-5)} \mathbb{1}_{[5,\infty)}(y)\, dy$

3. Density of Y: $\lambda e^{-\lambda(y-5)} \mathbb{1}_{[5,\infty)}(y)$