Rmk

$\sum_{x_1, x_2} p$ for discrete r. vector (X, Y)

p p.m.f

$\longleftrightarrow$

$\iint_{(x,y)} f$ for a continuous r. vector (X, Y)

f density

Bivariate density

$P(X < Y)$

$$= \int_{-\infty < x < y < \infty} f(x,y)\, dx\, dy$$

$$= 2\int_{-\infty < x < y < \infty} dx\, dy\; e^{-x} e^{-2y}\, \mathbb{1}_{(0,\infty)}(x)\, \mathbb{1}_{(0,y)}(y)$$

$$= 2\int_{0 < x < y < \infty} e^{-x} e^{-2y}\, dx\, dy$$

$$= 2\int_0^\infty dy\; e^{-2y} \Big(\overbrace{\int_0^y e^{-x}\, dx}^{= -e^{-x}\big|_0^y} \Big) = 1 - e^{-y}$$

$$= 2\int_0^\infty e^{-2y} (1 - e^{-y})\, dy$$

$$= 2\left\{ \int_0^\infty e^{-2y}\, dy - \int_0^\infty e^{-3y}\, dy \right\}$$

$$= 2\left\{ -\frac{e^{-2y}}{2}\Big|_0^\infty + \cancel{\int_0} \frac{e^{-3y}}{3}\Big|_0^\infty \right\}$$

$$= 2\left\{ \frac{1}{2} - \frac{1}{3} \right\} = \frac{2}{6} = \frac{1}{3}$$

Change of variable

Set $z = \frac{X}{Y}$. Let $\varphi \in C_b$. Then

1. $E[\varphi(Z)] = E[\varphi(\frac{X}{Y})]$ density of (X, Y)

$$= \int_{\mathbb{R}^2} \varphi\left(\frac{x}{y}\right) f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \varphi\left(\frac{x}{y}\right) e^{-(x+y)} \mathbb{1}_{(0,\infty)}(x)\, \mathbb{1}_{(0,\infty)}(y)\, dx\, dy$$

$$= \int_0^\infty \int_0^\infty \varphi\left(\frac{x}{y}\right) e^{-(x+y)}\, dx\, dy$$

2. CV in the plane: Bounds:

$z = \frac{x}{y}$, $\omega = y$ $\quad 0 < z, \omega < \infty$

Then $x = \omega z$ $\quad y = \omega$

$$J = \left| \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial \omega} & \frac{\partial y}{\partial \omega} \end{vmatrix} \right|$$ absolute value

$$= \left| \begin{vmatrix} \omega & 0 \\ z & 1 \end{vmatrix} \right|$$

$$= |\omega \times 1 - z \times 0| = |\omega| = \omega$$

Change of variable (ctd)

2 We get

$$E[\varphi(Z)]$$

$$= \int_0^\infty \int_0^\infty \varphi(z)\, \overbrace{e^{-(zw+w)}}^{e^{-w(z+1)}}\, w \, dz\, dw$$

$$= \int_0^\infty dz\, \frac{\varphi(z)}{(z+1)} \underbrace{\int_0^\infty w\, e^{-(z+1)w} \times (z+1)\, dw}_{\substack{E[W] \text{ with} \\ W \sim \mathcal{E}(z+1) \\ = \frac{1}{z+1}}}$$

$$= \int_0^\infty \varphi(z)\, \frac{1}{(z+1)^2}\, dz$$

$$= \int_{\mathbb{R}} \varphi(z)\, \underbrace{\frac{1}{(z+1)^2}\, \mathbb{1}_{(0,\infty)}(z)}_{\text{density of r.v. } Z}\, dz$$