Rmk on $\perp\!\!\!\perp$ of r.v.

If $A, B \subset \Omega$, then

$A \perp\!\!\!\perp B$ if $P(A \cap B) = P(A) P(B)$

For r.v., we say that $X \perp\!\!\!\perp Y$

if we have $A \perp\!\!\!\perp B$ where

$A = (X \in C)$

$B = (Y \in D)$

This should be true for all

$C, D \subset \mathbb{R}$

Rmk 2 It is usually tedious

to verify $P(X \in C, Y \in D)$
$= P(X \in C) \, P(Y \in D)$

for all sets C, D $\rightarrow$ other methods to check $X \perp\!\!\!\perp Y$

Coin tossing example

In order to get $X \perp\!\!\!\perp Y$, we should compare

$$\frac{1}{8} = P(X=(0,0)) = \overbrace{P(X_1=0)}^{1/2}\,\overbrace{P(X_2=0)}^{1/4}$$

$$\frac{3}{8} = P(X=(0,1)) = \overbrace{P(X_1=0)}^{1/2}\,\overbrace{P(X_2=1)}^{3/4}$$

$$\frac{1}{8} = P(X=(1,0)) = \overbrace{P(X_1=1)}^{1/2}\,\overbrace{P(X_2=0)}^{1/4}$$

$$\frac{3}{8} = P(X=(1,1)) = \overbrace{P(X_1=1)}^{1/2}\,\overbrace{P(X_2=1)}^{3/4}$$

We have obtained, thanks to Prop 10 - item (2), that

$$X_1 \perp\!\!\!\perp X_2$$

Rmk

$$\left.\begin{array}{l} X_1 = \mathbb{1}_A \\ X_2 = \mathbb{1}_B \\ A \perp\!\!\!\perp B \ (n=3) \end{array}\right\} \Rightarrow X_1 \perp\!\!\!\perp X_2$$

Romeo & Juliet

Model. If $X \sim \mathcal{U}([0,1])$, $f_X(x) = \mathbb{1}_{[0,1]}(x)$
$Y \sim \mathcal{U}([0,1])$, $f_Y(y) = \mathbb{1}_{[0,1]}(y)$
$X \perp\!\!\!\perp Y$

Then the joint density f for (X,Y)

is given by

$$f(x,y) = f_X(x)\, f_Y(y)$$
$$= \mathbb{1}_{[0,1]}(x)\, \mathbb{1}_{[0,1]}(y)$$
$$= \mathbb{1}_{[0,1]^2}(x,y)$$

We wish to compute

rescaled 10mn

$$P\left(|X-Y| \le \frac{1}{6} \right)$$ (diff of arrival times $\le$ 10mn)

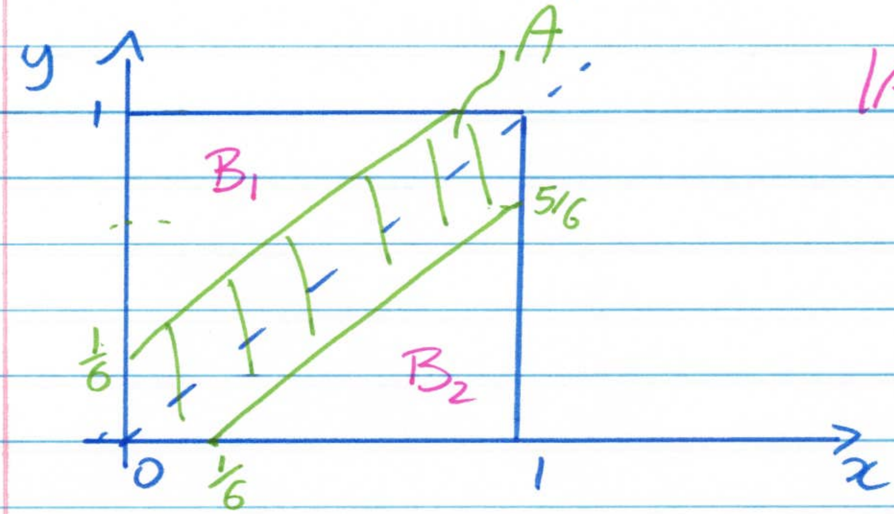
$$= \int_{|x-y| \le \frac{1}{6}} f(x,y)\, dx\, dy$$

$$= \text{Area}\left(\left\{ (x,y) \in [0,1]^2 ;\ |x-y| \le \frac{1}{6} \right\} \right)$$

Set

$$A = \{ (x,y) \in [0,1]^2 ; \quad |x-y| \leq \frac{1}{6} \}$$

We wish to compute Area $(A) = |A|$



We have

$$|A| = 1 - |B_1| - |B_2|$$

$$|B_1| = |B_2| = \frac{1}{2}\left(\frac{5}{6}\right)^2$$

Thus

$$|A| = 1 - \left(\frac{5}{6}\right)^2 \simeq 30.5\%$$

We get

$$P(R \text{ meets } J) \simeq 30.5\%$$

$\perp\!\!\!\perp$ Example (Prop 11)

(X, Y) has joint density

$$f(x,y) = 6\, e^{-(2x+3y)}\, \overbrace{\mathbb{1}_{(0,\infty)^2}(x,y)}^{= \mathbb{1}_{(0,\infty)}(x)\, \mathbb{1}_{(0,\infty)}(y)}$$

$$= \underbrace{6\, e^{-2x}\, \mathbb{1}_{(0,\infty)}(x)}_{h(x)}\ \underbrace{e^{-3y}\, \mathbb{1}_{(0,\infty)}(y)}_{g(y)}$$

Thus $X \perp\!\!\!\perp Y$

⚠ Here we don't claim that

h = marg. density of X
g = " " " Y

Check at home:

$$f_X(x) = 2\, e^{-2x}\, \mathbb{1}_{(0,\infty)}(x)$$
$$f_Y(y) = 3\, e^{-3x}\, \mathbb{1}_{(0,\infty)}(y)$$

Thus we have

$$X \sim \mathcal{E}(2), \quad Y \sim \mathcal{E}(3), \quad X \perp\!\!\!\perp Y$$

Non $\perp\!\!\!\perp$ example

Reason 1 for non $\perp\!\!\!\perp$: We have the constraint

$$0 \leq X + Y \leq 1$$

Thus if Y is close to 1, then

X is close to 0

(values of X depend on the values of Y)

Reason 2

$$f(x,y) = 24x \, \mathbb{1}_{(0,\infty)}(x)$$

$$\times \; y \, \mathbb{1}_{(0,\infty)}(y)$$

$$\times \; \mathbb{1}_{(0<x+y<1)}$$

not under product form

Non $\perp\!\!\!\perp$ example (2)

In order to prove that $X \not\perp\!\!\!\perp Y$, we show

$$P\left(X \leq \tfrac{1}{2},\ Y \leq \tfrac{1}{2}\right)$$

$$\neq$$

$$P\left(X \leq \tfrac{1}{2}\right) P\left(Y \leq \tfrac{1}{2}\right)$$

(ii) $P(X \le \frac{1}{2}, Y \le \frac{1}{2})$

$$= \int_{\mathbb{R}^2} \mathbb{1}_{(x \le \frac{1}{2},\, y \le \frac{1}{2})} f(x,y)\, dx\, dy$$

$$= \int_{\mathbb{R}^2} \mathbb{1}_{(x \le \frac{1}{2},\, y \le \frac{1}{2})} 24\, xy\, \mathbb{1}_{(0,\infty)^2}(x,y) \times \mathbb{1}_{(0 < x+y \le 1)}\, dx\, dy$$

Rmk: $x \le \frac{1}{2},\ y \le \frac{1}{2} \Rightarrow x+y \le 1$.
We get

$$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2})$$

$$= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} 24\, xy\, dx\, dy$$

$$= 24 \overbrace{\left(\int_0^{\frac{1}{2}} x\, dx \right)}^{= \frac{x^2}{2}\Big|_0^{\frac{1}{2}} = \frac{1}{8}} \overbrace{\left(\int_0^{\frac{1}{2}} y\, dy \right)}^{\to \frac{1}{8}}$$

$$= \frac{3}{8}$$

$$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = \frac{3}{8}$$

Non $\perp\!\!\!\perp$ example (3)

$P(X \leq \frac{1}{2})$

$$= 24 \int_{\mathbb{R}^2} \underbrace{\mathbb{1}_{(x \leq \frac{1}{2})} \mathbb{1}_{(0 \leq x \leq 1)}}_{\mathbb{1}_{(0 \leq x \leq \frac{1}{2})}} \mathbb{1}_{(0 \leq y \leq 1)} \mathbb{1}_{(0 \leq x+y \leq 1)} \, x \, y \, dx \, dy$$

$$= 24 \int_0^{\frac{1}{2}} dx \; x \left(\underbrace{\int_0^{1-x} y \, dy}_{= \frac{y^2}{2}\Big|_0^{1-x} = \frac{(1-x)^2}{2}} \right)$$

$$= \frac{24}{2} \int_0^{\frac{1}{2}} x \, (1-x)^2 \, dx$$

$$= 12 \int_0^{\frac{1}{2}} x \, (x^2 - 2x + 1) \, dx$$

$$= 12 \int_0^{\frac{1}{2}} (x^3 - 2x^2 + x) \, dx$$

$$= 12 \left\{ \frac{x^4}{4} - \frac{2}{3} x^3 + \frac{x^2}{2} \Big|_0^{\frac{1}{2}} \right\}$$

$$= \frac{11}{16} = P(Y \leq \frac{1}{2})$$

Non $\perp\!\!\!\perp$ example (4)

We have found

$$P(X \le \tfrac{1}{2}, Y \le \tfrac{1}{2}) = \frac{3}{8}$$

$$\neq$$

$$P(X \le \tfrac{1}{2})\, P(Y \le \tfrac{1}{2}) = \left(\frac{11}{16}\right)^2$$

Thus $\boxed{X \not\perp\!\!\!\perp Y}$