Meaning of $p_{X|Y}(x|y)$

$$p_{X|Y}(x,y) = P(\overbrace{X=x}^{A} \mid \overbrace{Y=y}^{B})$$

$$= \frac{\overbrace{P((X=x)\cap(Y=y))}^{p(x,y)}}{\underbrace{P(Y=y)}_{p_Y(y)}}$$

$$= \frac{p(x,y)}{p_Y(y)}$$

Cond prob. tossing 3 coins

$$p_{X_2|X_1}(0|0) = \frac{P(X_2=0, X_1=0)}{P(X_1=0)}$$

$$= \frac{p(0,0)}{p_{X_1}(0)}$$

$$= \frac{1/8}{1/2} = \frac{1}{4}$$

$$p_{X_2|X_1}(1|0) = \frac{P(X_2=1, X_1=0)}{P(X_1=0)}$$

$$= \frac{p(0,1)}{p_{X_1}(0)}$$

$$= \frac{3/8}{1/2} = \frac{3}{4}$$

Conclusion: $\mathcal{L}(X_2 \mid X_1=0)$

$$= \mathrm{Bin}(3/4)$$

Conditioning Poisson

Aim: $P(X=k \mid X+Y=n) \overset{(?)}{=} \binom{n}{k} p^k (1-p)^{n-k}$

We have

$$P(X=k \mid X+Y=n)$$

$$= \frac{P(X=k,\ X+Y=n)}{P(X+Y=n)}$$

$$= \frac{P(X=k,\ Y=n-k)}{P(X+Y=n)}$$

$$\overset{\perp\!\!\perp}{=} \frac{P(X=k)\ P(Y=n-k)}{P(X+Y=n)}$$

$X+Y \sim P(\lambda_1+\lambda_2)$

Conditioning Poisson (2). We get

$$P(X=k \mid X+Y=n)$$

$$= \frac{\cancel{e^{-\lambda_1}} \frac{\lambda_1^k}{k!} \; \cancel{e^{-\lambda_2}} \frac{\lambda_2^{n-k}}{(n-k)!}}{\cancel{e^{-(\lambda_1+\lambda_2)}} \frac{(\lambda_1+\lambda_2)^n}{n!}}$$

$$= \underbrace{\frac{n!}{k!\,(n-k)!}}_{\binom{n}{k}} \; \frac{\lambda_1^k \; \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^{n}} \quad n = k+(n-k)$$

$$= \binom{n}{k} \left(\underbrace{\frac{\lambda_1}{\lambda_1+\lambda_2}}_{p} \right)^k \left(\underbrace{\frac{\lambda_2}{\lambda_1+\lambda_2}}_{1-p} \right)^{n-k}$$

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Thus $\mathcal{L}(X \mid X+Y=n)$

$$= \text{Bin}(n, p)$$