Rmk about cdt densities

We also have

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)}$$

Heuristics about $f_{X|Y}$

$$f_{X|Y}(x \mid y)\,dx = \frac{f(x,y)}{f_Y(y)\,dy}\,dx\,dy$$

$$\simeq \frac{P(X \in [x, x+dx),\ Y \in [y, y+dy))}{P(Y \in [y, y+dy))}$$

$$= P(X \in [x, x+dx) \mid Y \in [y, y+dy))$$

Simple example

Step 1: Compute $f_Y(y)$.

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx$$

$$= \int_{\mathbb{R}} \frac{e^{-\frac{x}{y}}}{y}\, e^{-y}\, \mathbb{1}_{(0,\infty)}(x)\, \mathbb{1}_{(0,\infty)}(y)\,dx$$

$$= e^{-y}\, \mathbb{1}_{(0,\infty)}(y) \underbrace{\int_0^{\infty} \frac{e^{-\frac{x}{y}}}{y}\,dx}_{=1}$$

density of $\mathcal{E}(\frac{1}{y})$

$$= e^{-y}\, \mathbb{1}_{(0,\infty)}(y) \qquad \text{Thus } Y \sim \mathcal{E}(1)$$

Step 2: Compute $f_{X|Y}(x|y)$

$$= \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{\frac{e^{-x/y}}{y}\, \cancel{e^{-y}}\, \mathbb{1}_{(0,\infty)}(x)\, \cancel{\mathbb{1}_{(0,\infty)}(y)}}{\cancel{e^{-y}}\, \cancel{\mathbb{1}_{(0,\infty)}(y)}}$$

Step 2 (Ctd).

We get

$$f_{X|Y}(x|y) = \frac{1}{y} e^{-\frac{x}{y}} \mathbb{1}_{(0,\infty)}(x)$$

Conclusion: $\mathcal{L}(X \mid Y=y) = \mathcal{E}\left(\frac{1}{y}\right)$

Step 3: Compute

$\mathbb{P}(X > 1 \mid Y=y)$

$$= \int_1^\infty f_{X|Y}(x|y)\, dx$$

$$= \int_1^\infty \frac{1}{y} e^{-\frac{x}{y}}\, dy$$

$$= e^{-\frac{1}{y}}$$ (Tail of $\mathcal{E}\left(\frac{1}{y}\right)$)