Density of (X, Y):

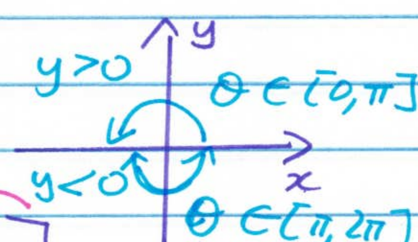
$$f_X(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \qquad f_Y(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

$X \perp\!\!\!\perp Y$

$$\Rightarrow f(x,y) = f_X(x)\, f_Y(y)$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{2}(x^2+y^2)\right)$$

Then in order to compute the density of (R, Θ), we consider $\varphi \in C_b(\mathbb{R}^2)$. We have



(1) $E[\varphi(R,\Theta)]$

$$= \overbrace{E[\varphi(R,\Theta)\, \mathbb{1}_{Y>0}]}^{A^+}$$

$$+ \underbrace{E[\varphi(R,\Theta)\, \mathbb{1}_{Y<0}]}_{A^-}$$

1 (Ctd)

$$A^+ = E[\varphi(R,\Theta)\, \mathbb{1}_{(Y>0)}]$$

$$= E[\varphi((X^2+Y^2)^{\frac{1}{2}}, \tan^{-1}(\tfrac{Y}{X}))\, \mathbb{1}_{(Y>0)}]$$

$$= \int_{\mathbb{R}^2} \varphi((x^2+y^2)^{\frac{1}{2}}, \tan^{-1}(\tfrac{y}{x}))\, \frac{e^{-\frac{1}{2}\overbrace{(x^2+y^2)}^{=r^2}}}{2\pi}\, dx\, dy$$

2 <u>CV: polar coordinates</u>

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

Then

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix}$$

$$= r\,(\cos^2(\theta) + \sin^2(\theta)) = r$$

Bounds: $r \in [0, \infty)$, $0 < \theta < \pi$

We get

$$A^+ = \int_0^\infty dr \int_0^\pi d\theta \; \varphi(r,\theta) \, \frac{e^{-\frac{1}{2}r^2}}{2\pi} \, r$$

J

In the same way

$$A^- = \int_0^\infty dr \int_\pi^{2\pi} d\theta \; \varphi(r,\theta) \, \frac{e^{-\frac{1}{2}r^2}}{2\pi} \, r$$

We end up with

$$E[\varphi(R,\Theta)] = A^+ + A^-$$

$$= \int_0^\infty dr \int_0^{2\pi} d\theta \; \varphi(r,\theta) \, \frac{e^{-\frac{1}{2}r^2}}{2\pi} \, r$$

$$= \int_{\mathbb{R}^2} \varphi(r,\theta) \; r e^{-\frac{1}{2}r^2} \mathbb{1}_{(0,\infty)}(r) \times \frac{\mathbb{1}_{[0,2\pi]}(\theta)}{2\pi} \; dr\, d\theta$$

$f_R(r)$

$f(r,\theta)$

density $f_\Theta(\theta)$

$\Theta \sim \mathcal{U}([0,2\pi])$