Computation of $E[X]$, $X \sim G(p)$

$$E[X] = \sum_{i=1}^{\infty} i \, (\overbrace{1-p}^{1-p=q})^{i-1} p$$

$$= \sum_{i=1}^{\infty} (i-1+1) \, q^{i-1} \, p$$

$$= \underbrace{\sum_{i=1}^{\infty} (i-1) \, q^{i-1} p}_{B} + \underbrace{\left(\sum_{i=1}^{\infty} q^{i-1} \right)}_{A} p$$

Then (geometric series, ratio $= q$)

$$A = \frac{1}{1-q} = \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$B = \left(\sum_{i=2}^{\infty} (i-1) \, q^{i-1} \right) p$$

cv: $i-1 = j$. We get

$$B = \left(\sum_{j=1}^{\infty} j \, q^{j} \right) p$$

$$= \sum_{j=1}^{\infty} j \, q^{j-1} \times q \times p = q \, E[X]$$

$= E[X]$

We have found

$$E[X] = B + A\,p$$

$$= q\,E[X] + \frac{1}{p} \times p$$

$$= q\,E[X] + 1$$

Thus $E[X]$ solves

$$E[X] = q\,E[X] + 1$$

$$\Leftrightarrow \quad (1-q)\,E[X] = 1$$

$$\Leftrightarrow \quad E[X] = \frac{1}{1-q}$$ and $q = 1-p$

$$\Leftrightarrow \quad \boxed{E[X] = \frac{1}{p}}$$

Let $X \sim G(p)$, $p = \frac{1}{6}$

Then

$$P(X=5) = (1-p)^{5-\cancel{x}1}\, p$$

$$= \left(\frac{5}{6}\right)^4 \times \frac{1}{6} \simeq 8\%$$

↳ Probability that we get a 6 for the 1st time on the 5th roll is .08

We also have

$$P(X \geq 7) = (1-p)^{7-1}$$

$$\boxed{P(X \geq 7) = \left(\frac{5}{6}\right)^6}$$

$$\simeq 33\%$$

Proof of Prop 12

$$P(X \geq n) = \sum_{j=n}^{\infty} P(X = j)$$

$$= \sum_{j=n}^{\infty} (1-p)^{j-1} p$$

$$= p \underbrace{\sum_{j=n}^{\infty} (1-p)^{j-1}}$$

geom series
ratio $= 1-p$
first term $= (1-p)^{n-1}$

$$= p \frac{(1-p)^{n-1}}{1-(1-p)}$$

$$= \cancel{p} \frac{(1-p)^{n-1}}{\cancel{p}}$$

$$= (1-p)^{n-1}$$

Prop 13 If $X \sim Nbin(r, p)$
$Y \sim Nbin(r+1, p)$

Then $E[X^l] = \frac{r}{p} E[(Y-1)^{l-1}]$

Hence, for $l=1$

$$E[X^l] = \frac{r}{p} E[\overbrace{(Y-1)^0}^{1}] = \frac{r}{p}$$

$$\Rightarrow \quad E[X] = \frac{r}{p}$$

For $l=2$

$$E[X^2] = \frac{r}{p} E[(Y-1)^1]$$

$$= \frac{r}{p} E[Y-1] \overset{\text{linearity}}{=} \frac{r}{p}(\overbrace{E[Y]}^{=\frac{r+1}{p}} - 1)$$

$$= \frac{r}{p}\left(\frac{r}{p} + \frac{1}{p} - 1\right)$$

$$= \frac{r^2}{p^2} + \frac{r}{p^2} - \frac{r}{p}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \cancel{\frac{r^2}{p^2}} + \frac{r}{p^2} - \frac{r}{p} - \cancel{\left(\frac{r}{p}\right)^2} = \frac{r(1-p)}{p^2}$$

Banach match problem

E_k = (Discover that rh empty & k matches in lh)

Experiment: pick 1 side at random

↳ Success if we pick rh , $p = \frac{1}{2}$

X = # trials in order to get (N+1) successes

↳ when discover that rh is empty

Then $X \sim Nbin(r = N+1, p = \frac{1}{2})$

If we have k matches left in the lh pocket, it means that

$$X = (N+1) + (N-k)$$

We have obtained

$$P(E_k) = P(X = \overbrace{(N+1) + (N-k)}^{j})$$

Summary. We have

$$P(E_k) = P(X = j) \quad , \quad j = 2N+1-k$$

$$= \binom{j-1}{\underbrace{r}_{=N+1}-1} p^r (1-p)^{j-r}$$

(formula for Nbin)

$$= \binom{2N-k}{N} \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{j-r}$$

$$= \binom{2N-k}{N} \left(\frac{1}{2}\right)^j$$

$$= \binom{2N-k}{N} \left(\frac{1}{2}\right)^{2N-k+1}$$

Moreover, by symmetry

P(one box empty, k matches in the other box)

$$= 2\, P(E_k)$$

$$= \binom{2N-k}{N} \times \cancel{2} \times \left(\frac{1}{2}\right)^{2N-k\cancel{+1}} = \binom{2N-k}{N} \left(\frac{1}{2}\right)^{2N-k}$$