Outline

Expectation of sums

Proposition 15.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X_1, \ldots, X_n : S \to \mathbb{R}$ n random variables

Hypothesis: S is countable, i.e

$$S = \{s_i; i \geq 1\}$$

Then

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i]$$

Example: number of successes (1)

Experiment:

- n trials
- Success for i-th trial with probability p_i
- $X = \#$ of successes

Question:
Expression for $\mathbf{E}[X]$ and $\cancel{\mathbf{Var}}(X)$

Situation: we have n trials

For $i = 1, \dots, n$ we set

$$X_i = \begin{cases} 1 & \text{if success } i\text{-th trial} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{P}(A_i) = p_i$

Otherwise stated, if A_i = "success ith trial" then

$$X_i = \mathbb{1}_{A_i} \Rightarrow X_i \sim B(p_i)$$

$X_i = \mathbb{1}_{A_i} = \mathbb{1}(\text{success , ith trial})$

Now

$$X \equiv \text{" number of success "}$$

$$= \sum_{i=1}^{n} X_i$$

$$\Rightarrow \boxed{\mathbb{E}[X]} = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$$

Prop 15

$$= \sum_{i=1}^{n} \overbrace{\mathbb{E}[X_i]}^{=p_i}$$

$$\boxed{= \sum_{i=1}^{n} p_i}$$

Particular case when p_i does not depend on i

$$p_1 = p_2 = \cdots = p_n = p$$

Then each $X_i \sim \mathcal{B}(p)$

and $X = \sum_{i=1}^{n} X_i \sim \text{Bin}(n, p)$

We get

$$\boxed{\mathbb{E}[X]} = \sum_{i=1}^{n} \overbrace{\mathbb{E}[X_i]}^{p} = \boxed{n\,p}$$

↳ easy way to find $\mathbb{E}[X]$ if $X \sim \text{Bin}(n,p)$

Example: number of successes (2)

Expression for X: Let

$$X_i = \mathbf{1}_{(\text{success for } i\text{-th trial})}$$

Then

$$X = \sum_{i=1}^{n} X_i$$

Expression for $\mathbf{E}[X]$: Thanks to Proposition 15, we have

$$\mathbf{E}[X] = \sum_{i=1}^{n} p_i$$

Outline

cdf

$$F(x) = F_X(x) = \mathbb{P}(X \leq x)$$

Continuity of the cdf

Proposition 16.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$
- F the cdf of X, i.e $F(x) = \mathbf{P}(X \leq x)$

Then the function F satisfies

1. F is a nondecreasing function
2. $\lim_{b\to\infty} F(b) = 1$
3. $\lim_{b\to-\infty} F(b) = 0$
4. F is right continuous

Proof of item 1

Inclusion property: Let $a < b$. Then

$$(X \le a) \subset (X \le b)$$

Consequence on probabilities:

$$\mathbf{P}(X \le a) \le \mathbf{P}(X \le b)$$

Proof of item 2

Definition of an increasing sequence: Let $b_n \nearrow \infty$ and

$$E_n = (X \leq b_n)$$

Then

$$\lim_{n\to\infty} E_n = (X < \infty)$$

Consequence on probabilities:

$$\begin{aligned} 1 &= \mathbf{P}(X < \infty) \\ &= \mathbf{P}\left(\lim_{n\to\infty} E_n\right) \\ &= \lim_{n\to\infty} \mathbf{P}(E_n) \quad \text{(Since } n \mapsto E_n \text{ is increasing)} \\ &= \lim_{n\to\infty} F(b_n) \end{aligned}$$

Example of cdf (1)

F non decreasing $\lim_{x\to\infty} F(x) = 1$

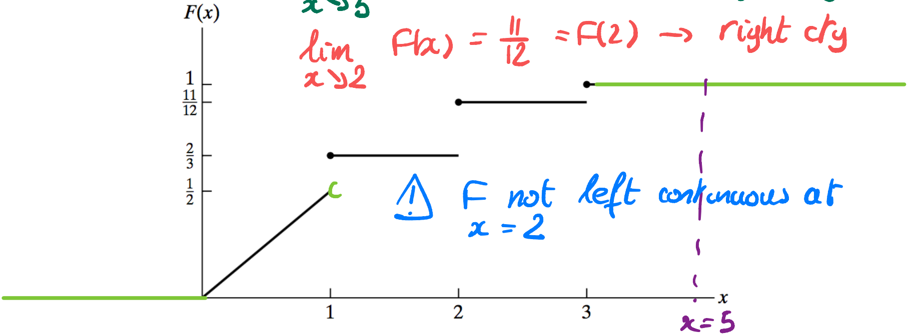
$\lim_{x\to-\infty} F(x) = 0$

Definition of the function: We set

$$F(x) = \frac{x}{2}\,\mathbf{1}_{[0,1)}(x) + \frac{2}{3}\,\mathbf{1}_{[1,2)}(x) + \frac{11}{12}\,\mathbf{1}_{[2,3)}(x) + \mathbf{1}_{[3,\infty)}(x)$$

$\lim_{x\searrow 5} F(x) = 1 = F(5) \to$ right cty

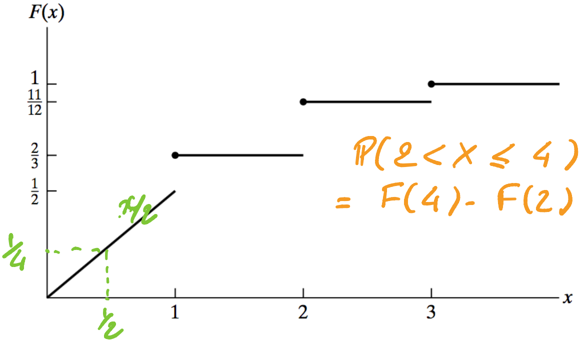
$\lim_{x\searrow 2} F(x) = \frac{11}{12} = F(2) \to$ right cty



⚠ F not left continuous at $x = 2$

Example of cdf (2) F cdf of a r.v. X

Some information read on the graph (see next page):



$\mathbb{P}(2 < X \leq 4)$
$= F(4) - F(2) = 1 - \frac{11}{12} = \frac{1}{12}$

$\mathbb{P}(X > \frac{1}{2}) = 1 - \mathbb{P}(X \leq \frac{1}{2}) = 1 - F(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4}$

$\mathbb{P}(X < 3) = \mathbb{P}(X \leq 3) - \mathbb{P}(X = 3) = F(3-) = \frac{11}{12}$

$\mathbb{P}(X = 1) = F(1) - F(1-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

Example of cdf (3)

Information read on the cdf: One can check that

- $\mathbf{P}(X < 3) = \frac{11}{12}$
- $\mathbf{P}(X = 1) = \frac{1}{6}$
- $\mathbf{P}(X > \frac{1}{2}) = \frac{3}{4}$
- $\mathbf{P}(2 < X \leq 4) = \frac{1}{12}$