Outline

1 Joint distribution functions

2 Independent random variables

3 Sums of independent random variables

4 Conditional distributions: discrete case

5 Conditional distributions: continuous case

6 Joint probability distribution of functions of random variables

7 Conditional expectation

Cond. pmf in the discrete case (repeated)

cond expectation: expectation wrt to cond pmf $p_{X|Y}$

Definition 24.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y
- y such that $p_Y(y) > 0$

Then the conditional pmf of X given $Y = y$ is defined by

$$p_{X|Y}(x|\, y) = \mathbf{P}\left(X = x|\, Y = y\right) = \frac{p(x, y)}{p_Y(y)}$$

Cond. expectation in the discrete case

Average value of X if we know that Y=y

Definition 25.

Let

- (X, Y) couple of discrete random variables
- Joint pmf p
- Marginal pmf's p_X, p_Y, y such that $p_Y(y) > 0$
- $p_{X|Y}(x|\,y)$ conditional distribution

Then the conditional exp. of X given $Y = y$ is defined by

$$\mathbf{E}\left[X|\, Y = y\right] = \sum_{x \in \mathcal{E}} x\, p_{X|Y}(x|\, y)$$

Binomial example (1)

Situation: Let

- $X, Y \sim \text{Bin}(n, p)$, $X \perp\!\!\!\perp Y$
- $Z = X + Y$

Problem: We wish to compute (Here m is a fixed value in $\{0, \dots, 2n\}$)

$$\mathbf{E}[X \mid Z = m]$$

<u>Situation</u>: $X \sim Bin(n,p)$ $\quad X \perp\!\!\!\perp Y$
$Y \sim Bin(n,p)$

Set $Z = X + Y$. Then $Z \sim Bin(2n, p)$

Take $m \in \{0, \dots, 2n\}$. Then

$$\boxed{p_{X|Z}(k \mid m)} = \mathbb{P}(X = k \mid Z = m)$$

$$= \frac{\mathbb{P}(X = k, \overset{X+Y}{Z} = m)}{\mathbb{P}(Z = m)}$$

$$= \frac{\mathbb{P}(X = k, Y = m - k)}{\mathbb{P}(Z = m)} \overset{\perp\!\!\!\perp}{=} \boxed{\frac{\mathbb{P}(X = k)\,\mathbb{P}(Y = m-k)}{\mathbb{P}(Z = m)}}$$

Cond. pmf

$$\boxed{P_{X|Z}(k \mid m)} = \frac{\overbrace{\mathbb{P}(X=k)}^{\mathrm{Bin}(n,p)}\ \overbrace{\mathbb{P}(Y=m-k)}^{\mathrm{Bin}(n,p)}}{\underbrace{\mathbb{P}(Z=m)}_{\mathrm{Bin}(2n,p)}}$$

$$= \frac{\binom{n}{k} \cancel{p^{k}} \cancel{(1-p)^{n-k}} \binom{n}{m-k} \cancel{p^{m-k}} \cancel{(1-p)^{n-m+k}}}{\binom{2n}{m} \cancel{p^{m}} \cancel{(1-p)^{2n-m}}}$$

$$\boxed{= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}}$$

Binomial example (2)

Distribution for Z:

$$Z = \sum_{i=1}^{n} X_i + \sum_{j=1}^{n} Y_j \sim \text{Bin}(2n, p)$$

Computation for conditional pmf: For $k \leq \min(n, m)$ we have

$$\begin{aligned} \mathbf{P}\left(X = k \middle| Z = m\right) &= \frac{\mathbf{P}(X = k,\, X + Y = m)}{\mathbf{P}(Z = m)} \\ &= \frac{\mathbf{P}(X = k,\, Y = m - k)}{\mathbf{P}(Z = m)} \\ &= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}} \end{aligned}$$

Binomial example (3)

Conditional pmf: For $k \leq \min(n, m)$ we have

$$p_{X|Z}(k|\, m) = \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

Recall: If $V \sim \mathrm{HypG}(n, N, m)$ then

$$\mathbf{P}(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

Identification of the conditional pmf: We have

$$p_{X|Z}(k|\, m) = \text{Pmf of HypG}(2n, m, n)$$

Summary:

gen case: $n \quad N \quad m$

$$p_{X|Z}(\cdot \mid m) \equiv \text{pmf of } HypG(2n, m, n)$$

Cond. expectation

$$\boxed{E[X \mid Z=m]} = \sum_{x=0}^{n} x \; p_{X|Z}(x \mid m)$$

$= \text{expected value of } HypG(2n, m, n)$

formula

$$= m \times \frac{n}{2n} \quad \boxed{= \frac{m}{2}}$$

Binomial example (4)

Conditional expectation: Let $V \sim \text{HypG}(2n, m, n)$. Then

$$\mathbf{E}\left[X \middle| Z = m\right] = \mathbf{E}[V]$$

Numerical value:
According to the values for hypergeometric distributions,

$$\mathbf{E}\left[X \middle| Z = m\right] = m \times \frac{n}{2n} = \frac{m}{2}$$