

# Outline

- 1 Weak law of large numbers
- 2 Central limit theorem

Toss a coin  $n = 3000$

$X = \#$  heads

We expect  $X$  to satisfy

$$X \approx 1500$$

Law large numbers : Indeed,

$$X \approx 1500$$

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1 Weak law of large numbers

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# Weak law of large numbers

$x_i$ 's are i.i.d  $\Rightarrow E[X_i] = E[X_1] = \mu$

## Theorem 1.

Consider

- A sequence  $\{X_i; i \geq 1\}$  of i.i.d random variables
- Write  $E[X_i] = \mu$
- Set

empirical mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Then for any  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \mathbf{P} (|\bar{X}_n - \mu| > \varepsilon) = 0$$

$\bar{X}_n \approx \mu$

## Coin tossing example

$X_i = 1$  (head,  $i$ th toss)

Then  $X_i \sim B(1/2)$

$X_i$ 's iid with common dist  $B(1/2)$

We get, if  $n$  large

$$\bar{X}_n = \frac{\sum X_i}{n} \approx 0.5 \quad (\text{LLN})$$

If  $n = 3000$ , we get

$$\sum X_i \approx 3000 \times 0.5 = 1500$$

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# DeMoivre-Laplace theorem (repeated)

## Theorem 2.

Let

- $n \geq 1, p \in (0, 1)$
- $X_n \sim \text{Bin}(n, p)$
- $a < b$

Then

$$\lim_{n \rightarrow \infty} \mathbf{P} \left( a < \frac{X_n - np}{(np(1-p))^{1/2}} < b \right) = \Phi(b) - \Phi(a)$$

$X_n = \sum_{i=1}^n Y_i$ . We have to  
change the variable  
 $X_n \rightarrow \bar{X}_n = \frac{X_n}{n}$

# Another way to write De Moivre's theorem

## Theorem 3.

Consider

- A sequence  $\{Y_i; i \geq 1\}$  of indep.  $\mathcal{B}(p)$  random variables
- We have  $\mathbf{E}[Y_i] = p$  and  $\mathbf{Var}(Y_i) = p(1 - p) \equiv \sigma^2$
- Set

$$\bar{Y}_n = \frac{1}{n} \sum_{i=1}^n Y_i$$

Then for any  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \mathbf{P} \left( a < \sqrt{n} \left( \frac{\bar{Y}_n - p}{\sigma} \right) < b \right) = \Phi(b) - \Phi(a)$$

# Central limit theorem

## Theorem 4.

Consider

- A sequence  $\{X_i; i \geq 1\}$  of i.i.d random variables
- Write  $\mathbf{E}[X_i] = \mu$  and  $\mathbf{Var}(X_i) = \sigma^2$
- Set

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Then for any  $\varepsilon > 0$  we have

$$\lim_{n \rightarrow \infty} \mathbf{P} \left( a < \sqrt{n} \left( \frac{\bar{X}_n - \mu}{\sigma} \right) < b \right) = \Phi(b) - \Phi(a)$$

Remark: We have seen  $\bar{X}_n \approx \mu$   
Now we have

$$\sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} \approx z \quad \rightarrow z \sim \mathcal{N}(0,1)$$

$$\Rightarrow \bar{X}_n - \mu \approx \frac{\sigma z}{\sqrt{n}}$$

$$\Rightarrow \bar{X}_n \approx \mu + \frac{\sigma z}{\sqrt{n}}$$

random fluctuations  
of order  $\frac{1}{\sqrt{n}}$

$\bar{X}_n$  due to  $\mu = E[X_i]$

## Problem 8.5 (1)

**Situation:** We have

- Fifty numbers rounded off to the nearest integer and then summed
- The individual round-off errors are uniformly distributed over  $(-0.5, 0.5)$

**Question:**

Approximate the probability that the resultant sum differs from the exact sum by more than 3.

Model :  $X_i =$  error for  $i$ -th number

Hyp  $X_i$ 's are iid  $X_i \sim U([-0.5, 0.5])$

We want to evaluate

$$P\left( \left| \sum_{i=1}^n X_i \right| \geq 3 \right)$$

$$\sigma^2 = \frac{1}{12} \quad \mu = 0$$
$$n = 50$$

$$= P\left( \left| \frac{1}{n} \sum_{i=1}^n X_i \right| \geq \frac{3}{n} \right)$$

$$= P\left( |\bar{X}_n| \geq \frac{3}{n} \right)$$

$$= P\left( \left| \sqrt{n} \left( \frac{\bar{X}_n - \mu}{\sigma} \right) \right| \geq \frac{3/n - 0}{\sigma} \right)$$

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$$P\left( |Z| \geq \frac{3/50}{\sqrt{12}} \right) \stackrel{\text{Table}}{=} 14\%$$

## Problem 8.5 (2)

Model: Set

$X_i \equiv i$ -th error

We wish to find

$$\mathbf{P} \left( \left| \sum_{i=1}^n X_i \right| > 3 \right)$$

Law of  $X_i$ : We have

- $X_i$ 's i.i.d
- $X_i \sim \mathcal{U}([-0.5, 0.5])$
- $\mathbf{E}[X_i] = 0$  , and  $\mathbf{Var}(X_i) = \sigma^2 = \frac{1}{12}$

## Problem 8.5 (3)

Application of CLT: Write

$$\begin{aligned} \mathbf{P} \left( \left| \sum_{i=1}^n X_i \right| > 3 \right) &= \mathbf{P} \left( \left| \sqrt{n} \bar{X}_n \right| > \frac{3}{\sqrt{50}} \right) \\ &= \mathbf{P} \left( \left| \sqrt{n} \frac{\bar{X}_n}{\sigma} \right| > \frac{3\sqrt{12}}{\sqrt{50}} \right) \\ &\stackrel{CLT}{\approx} \mathbf{P} (|Z| > 1.47) \\ &= .14 \end{aligned}$$