Final Fall 24 - Solutions

Problem 1. A student is getting ready to take an important oral examination. There are $n = 3$ examiners. We assume for now that the student has a "Good" day. This means that each of his examiners will pass him, independently of one another, with probability $p_g = 0.8$.

1.1. For $i = 1, 2, 3$, we set

$$X_i = \mathbf{1}_{\{\text{Examiner } i \text{ passes the student}\}}, \quad \text{and} \quad X = \text{Number of examiners passing the student}$$

Express X as a function of the X_i's.

$$\boxed{X = \sum_{i=1}^{3} X_i}$$

1.2. Identify the laws of X_i and X.

We have

(i) $\boxed{X_i \sim \mathcal{B}(p_g) \quad , \quad \text{with } p_g = 0.8}$

(ii) Since

- $X = \sum_{i=1}^{n} X_i$
- X_i i.i.d with law $\mathcal{B}(p_g)$

we have

$$\boxed{\begin{array}{l} X \sim \text{Bin}(n, p_g) \\ \text{with } n = 3 , \; p_g = 0.8 \end{array}}$$

1.3. The student will pass the examination if a majority of the examiners pass him. Let us call A the event "Student passes the examination". Compute $\mathbf{P}(A)$ for a good day.

We have

$$A = (X \geq 2)$$

Thus

$$\mathbb{P}(A) = \mathbb{P}(X \geq 2)$$

$$= \mathbb{P}(X=2) + \mathbb{P}(X=3)$$

$$= \binom{n}{2} p_g^2 (1-p_g) + \binom{n}{3} p_g^3$$

$$= \binom{3}{2} \times (0.8)^2 \times 0.2 + \binom{3}{3} \times (0.8)^3$$

$$= 3 \times (0.8)^2 \times 0.2 + (0.8)^3$$

$$= 0.8^2 (3 \times 0.2 + 0.8)$$

$$= 0.64 \times 1.4$$

$$\boxed{\mathbb{P}(A) = 0.896}$$

1.4. We now assume that (1) The students also has "Bad" days, for which the probability that each of his examiners will pass him becomes $p_b = 0.4$. (2) If G (resp. B) denotes the event "Good day" (resp. "Bad day"), then $\mathbf{P}(G) = \frac{2}{3}$ (resp. $\mathbf{P}(B) = \frac{1}{3}$). By writing a proper conditioning, compute $\mathbf{P}(A)$ in this new situation.

(i) What we have computed in 1-3 is in fact

$$\mathbb{P}(A \mid G) = \binom{n}{2} p_g^2 (1-p_g) + \binom{n}{3} p_g^3$$

$$\boxed{\mathbb{P}(A \mid G) = 0.896}$$

(ii) In the same way, we have

$$\mathbb{P}(A \mid B) = \binom{n}{2} p_b^2 (1-p_b) + \binom{n}{3} p_b^3$$

$$= 3 \times (0.4)^2 \times 0.6 + (0.4)^3$$

$$= 0.16 \times (3 \times 0.6 + 0.4)$$

$$= 0.16 \times 2.2$$

$$\boxed{\mathbb{P}(A \mid B) = 0.352}$$

(iii) According to Bayes 1:

$$\mathbb{P}(A) = \mathbb{P}(A|G)\,\mathbb{P}(G) + \mathbb{P}(A|B)\,\mathbb{P}(B)$$

$$= 0.896 \times \frac{2}{3} + 0.352 \times \frac{1}{3}$$

$$\boxed{\mathbb{P}(A) = 0.715}$$

Problem 2. The time (in hours) required to repair a machine is an exponentially distributed random variable X with parameter $\lambda = \frac{1}{2}$.

2.1. Compute the value of $\mathbf{E}[X]$ for a random variable $X \sim \mathcal{E}(\lambda)$ with a general $\lambda > 0$. You should not apply the formula directly, you are asked to calculate the corresponding integral.

$$\mathbb{E}[X] = \int_0^\infty \underbrace{x}_{u} \, \underbrace{\lambda e^{-\lambda x}}_{v'} \, dx$$

We have $u' = 1$, $v = -e^{-\lambda x}$.
Thus by integration by parts,

$$\mathbb{E}[X] = -x e^{-\lambda x} \Big|_0^\infty + \int_0^\infty e^{-\lambda x} \, dx$$

$$= 0 - \frac{1}{\lambda} e^{-\lambda x} \Big|_0^\infty$$

$$\boxed{\mathbb{E}[X] = \frac{1}{\lambda}}$$

2.2. What is the probability that a repair time exceeds 2 hours?

$$\mathbb{P}(X > 2) = \int_2^\infty \lambda e^{-\lambda x}\, dx$$

$$= -e^{-\lambda x}\Big|_2^\infty$$

$$= e^{-2\lambda}$$

Here $\lambda = \frac{1}{2}$. Thus

$$\mathbb{P}(X > 2) = e^{-1}$$

$$\boxed{\mathbb{P}(X>2) \simeq 0.37}$$

Note: One could also apply directly the formula for tails of $\mathcal{E}(\lambda)$ r.v seen in class.

2.3. What is the conditional probability that a repair takes at least 11 hours, given that its duration exceeds 9 hours?

We wish to compute

$$\mathbb{P}(X > 11 \mid X > 9)$$

By the memoryless property of $\mathcal{E}(\lambda)$, we have

$$\mathbb{P}(X > 11 \mid X > 9)$$

$$= \mathbb{P}(X > 2)$$

Thus

$$\boxed{\mathbb{P}(X > 11 \mid X > 9) \simeq 0.37}$$

Problem 3. We have fifty numbers rounded off to the nearest integer and then summed. We call X_i the i-th round-off error. The random variables X_i are i.i.d with common distribution $\mathcal{U}([-0.5, 0.5])$.

3.1. Compute $\mathbf{E}[X_i]$. You should not apply the formula directly, you are asked to calculate the corresponding integral.

The density of X_i is

$$f(x) = \mathbb{1}_{[-0.5,0.5]}(x)$$

Thus

$$\mathbb{E}[X_i] = \int_{-0.5}^{0.5} x \, dx$$

$$= 0 \quad \text{(by symmetry)}$$

Thus

$$\boxed{\mathbb{E}[X_i] = 0}$$

3.2. Compute $\mathbf{Var}(X_i)$. You should not apply the formula directly, you are asked to calculate the corresponding integral.

Since $E[X_i]=0$, we have

$$Var(X_i) = E[X_i^2]$$

$$= \int_{-0.5}^{0.5} x^2 \, dx$$

$$= 2\int_0^{0.5} x^2 \, dx$$

$$= \frac{2}{3} x^3 \Big|_0^{0.5}$$

$$= \frac{2}{3} \times \frac{1}{2^3}$$

$$\boxed{Var(X_i) = \frac{1}{12}}$$

3.3. Consider the sum of the errors, $S = \sum_{i=1}^{n} X_i$. Approximate the probability that S is larger than 3.

Define $\bar{X}_n = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then

$$\mathbb{P}(|S| > 3)$$

$$= \mathbb{P}\left(|\bar{X}_n| > \frac{3}{n}\right)$$

$$= 1 - \mathbb{P}\left(-\frac{3}{n} < \bar{X}_n < \frac{3}{n}\right)$$

with $\sigma = \frac{1}{\sqrt{12}}$

$$= 1 - \mathbb{P}\left(-\sqrt{n}\frac{(\frac{3}{n} - 0)}{\sigma} < \sqrt{n}\frac{(\bar{X}_n - 0)}{\sigma} < \sqrt{n}\frac{\frac{3}{n} - 0}{\sigma}\right)$$

CLT

$$\simeq 1 - \mathbb{P}\left(-\frac{3}{\sqrt{n}\,\sigma} < Z < \frac{3}{\sqrt{n}\,\sigma}\right)$$

$Z \sim \mathcal{N}(0,1)$

$$= 2\,\mathbb{P}\left(Z > \frac{3}{\sqrt{n}\,\sigma}\right)$$

$$= 2\left(1 - \phi\left(\frac{3}{\sqrt{n}\,\sigma}\right)\right)$$

$$= 2\left(1 - \phi\left(3\left(\frac{12}{50}\right)^{\frac{1}{2}}\right)\right)$$

(or $\sigma \simeq 0.29$)

$$= 2\left(1 - \phi(1.47)\right)$$

$$\simeq 2\,(1 - 0.93)$$

$$\boxed{\mathbb{P}(|S| > 3) \simeq 0.14}$$