MA/STAT 416 Fall 2024
Probability Theory

Final

- Write on one side of each page only.
- You can use a calculator.
- A 4 pages long handwritten cheat sheet is allowed. It should only contain formulae and theorems (no example, no solved problem).
- You have 60 minutes.
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- GOOD LUCK!

Name:

Purdue ID:

Problem 1. A student is getting ready to take an important oral examination. There are $n = 3$ examiners. We assume for now that the student has a "Good" day. This means that each of his examiners will pass him, independently of one another, with probability $p_g = 0.8$.

1.1. For $i = 1, 2, 3$, we set

$$X_i = \mathbf{1}_{\{\text{Examiner } i \text{ passes the student}\}}, \quad \text{and} \quad X = \text{Number of examiners passing the student}$$

Express X as a function of the X_i's.

Solution:

1.2. Identify the laws of X_i and X.

Solution:

1.3. The student will pass the examination if a majority of the examiners pass him. Let us call A the event "Student passes the examination". Compute $\mathbf{P}(A)$ for a good day.

Solution:

1.4. We now assume that (1) The students also has "Bad" days, for which the probability that each of his examiners will pass him becomes $p_b = 0.4$. (2) If G (resp. B) denotes the event "Good day" (resp. "Bad day"), then $\mathbf{P}(G) = \frac{2}{3}$ (resp. $\mathbf{P}(B) = \frac{1}{3}$). By writing a proper conditioning, compute $\mathbf{P}(A)$ in this new situation.

Solution:

Problem 2. The time (in hours) required to repair a machine is an exponentially distributed random variable X with parameter $\lambda = \frac{1}{2}$.

2.1. Compute the value of $\mathbf{E}[X]$ for a random variable $X \sim \mathcal{E}(\lambda)$ with a general $\lambda > 0$. You should not apply the formula directly, you are asked to calculate the corresponding integral.

Solution:

2.2. What is the probability that a repair time exceeds 2 hours?

Solution:

2.3. What is the conditional probability that a repair takes at least 11 hours, given that its duration exceeds 9 hours?

Solution:

Problem 3. We have $n = 50$ numbers rounded off to the nearest integer and then summed. We call X_i the i-th round-off error. The random variables X_i are i.i.d with common distribution $\mathcal{U}([-0.5, 0.5])$.

3.1. Compute $\mathbf{E}[X_i]$. You should not apply the formula directly, you are asked to calculate the corresponding integral.

Solution:

3.2. Compute $\mathbf{Var}(X_i)$. You should not apply the formula directly, you are asked to calculate the corresponding integral.

Solution:

3.3. Consider the sum of the errors, $S = \sum_{i=1}^{n} X_i$. Approximate the probability that $|S|$ is larger than 3.

Solution:

Values of $\Phi(x)$ for some $x \geq 0$

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936