Combinatorial analysis

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Introduction to Probability Theory - MA 519

Mostly taken from *A first course in probability* by S. Ross

Outline

Outline

A simple example of counting

A communication system:

- Setup: n antennas lined up
- Functional system:
  $\hookrightarrow$ when no 2 consecutive defective antennas
- We know that m antennas are defective

Problem: compute

$$\mathbf{P}\,(\text{functional system})$$

A simple example of counting (2)

Particular instance of the previous situation:

- Take $n = 4$ and $m = 2$
- Possible configurations:

$$\begin{matrix} 0011 & 0101 & 0110 \\ 1001 & 1010 & 1100 \end{matrix}$$

- We get 3 working configurations among 6, and thus

$$\mathbf{P}\left(\text{functional system}\right) = \frac{1}{2}$$

Conclusion: need an effective way to count, that is

Combinatorial analysis

Outline

Basic principle of counting

Theorem 1.

Suppose 2 experiments to be performed and

- For Experiment 1, we have m possible outcomes
- For each outcome of Experiment 1
  $\hookrightarrow$ We have n outcomes for Experiment 2

Then

Total number of possible outcomes is $m \times n$

Proof

Sketch of the proof: Set

$(i, j) \equiv$ Outcome i for Experiment 1 & Outcome j for Experiment 2

Then enumerate possibilities

Application of basic principle of counting

Example: Small community with

- 10 women
- Each woman has 3 chidren

We have to pick one pair as mother & child of the year

Question:

How many possibilities?

Generalized principle of counting

Theorem 2.

Suppose r experiments to be performed and

- For Experiment 1, we have n_1 possible outcomes
- For each outcome of Experiment i
  $\hookrightarrow$ We have n_{i+1} outcomes for Experiment $i+1$

Then total number of possible outcomes is

$$\prod_{i=1}^{r} n_i = n_1 \times n_2 \times \cdots \times n_r$$

Application of basic principle of counting

Example 1: Find # possible 7 place license plates if

- First 3 places are letters
- Final 4 places are numbers

Answer: 175,760,000

Example 2: Find # possible 7 place license plates if

- First 3 places are letters
- Final 4 places are numbers
- No repetition among letters or numbers

Answer: 78,624,000

Outline

Permutations

Definition:

A permutation of n objects is an ordered sequence of those n objects.

Property:

Two permutations only differ according to the order of the objects

Counting:

Let P_n be the number of permutations for n objects. Then

$$P_n = n! = n \times (n-1) \cdots \times 2 = \prod_{j=1}^{n} j$$

Example of permutation

Example: 3 balls, Red, Black, Green

Permutations: RBG, RGB, BRG, BGR, GBN, GBR
$\hookrightarrow$ 6 possibilities

Formula: $P_3 = 3! = 6$

Proof for the counting number P_n

Sketch of the proof:

Direct application of Theorem 2

Example of permutation (1)

Problem:

Count possible arrangements of letters in PEPPER

Example of permutation (2)

Solution 1:

Consider all letters as distinct objects

$$P_1\ E_1\ P_2\ P_3\ E_2\ R$$

Then

$$P_6 = 6! = 720 \text{ possibilities}$$

Example of permutation (3)

Solution 2:
Do not distinguish P's and E's.

Then

$$\frac{P_6}{P_3\, P_2} = \frac{6!}{3!\, 2!} = 60 \text{ possibilities}$$

Outline

Combinations

Definition:

A combination of p objects among n objects is non ordered subset of p objects.

Property:

Two combinations only differ according to nature of their objects

Counting:

The number of combinations of p objects among n objects is

$$\binom{n}{p} = \frac{n!}{p!\,(n-p)!}$$

Proof of counting

Combination when order is relevant:
Number of possibilities is

$$n \times (n-1) \cdots \times (n-p+1) = \frac{n!}{(n-p)!}$$

Combination when order is irrelevant:
We divide by # permutations of p objects
Number of possibilities is

$$\frac{n \times (n-1) \cdots \times (n-p+1)}{p!} = \frac{n!}{p!(n-p)!} = \binom{n}{p}$$

Example of combination (1)

Situation:

- We have a group of 5 women and 7 men
- We wish to form a committee with 2 women and 3 men

Problem:

- Find the number of possibilities

Example of combination (2)

Number of possibilities:

$$\binom{5}{2}\binom{7}{3} = 350$$

Example of combination (3)

Situation 2:

- We have a group of 5 women and 7 men
- We wish to form a committee with 2 women and 3 men
- 2 men refuse to serve together

Problem:

- Find the number of possibilities

Example of combination (4)

New number of possibilities:

$$\binom{5}{2}\left\{\binom{7}{3}-\binom{2}{2}\binom{5}{1}\right\}=300$$

Binomial theorem

Theorem 3.

Let

- $x_1, x_2 \in \mathbb{R}$
- $n \geq 1$

Then

$$(x_1 + x_2)^n = \sum_{k=0}^{n} \binom{n}{k} x_1^k \, x_2^{n-k}$$

Combinatorial proof

First expansion:

$$(x_1 + x_2)^n = \sum_{(i_1,\ldots,i_n)\in\{1,2\}^n} x_{i_1}\, x_{i_2} \cdots x_{i_n}$$

Definition of a family of sets:

$$A_k = \{(i_1, \ldots, i_n) \in \{1,2\}^n;\ \text{there are } k\ j\text{'s such that } i_j = 1\}\,.$$

New expansion: we have (convention: $|A_k| \equiv \mathrm{Card}(A_k)$)

$$\begin{aligned}(x_1 + x_2)^n &= \sum_{k=0}^{n} |A_k|\, x_1^k\, x_2^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x_1^k\, x_2^{n-k}\end{aligned}$$

Application of the binomial theorem

Proposition 4.

Let

- A a set with $|A| = n$
- $\mathcal{P}_n \equiv$ collection of all subsets of A

Then

$$|\mathcal{P}_n| = 2^n$$

Proof

Decomposition of $|\mathcal{P}_n|$: Write

$$
\begin{aligned}
|\mathcal{P}_n| &= \sum_{k=0}^{n} |\text{Subsets of } A \text{ with } k \text{ elements}| \\
&= \sum_{k=0}^{n} \binom{n}{k}
\end{aligned}
$$

Application of the binomial theorem:

$$
\begin{aligned}
|\mathcal{P}_n| &= (1+1)^n \\
&= 2^n
\end{aligned}
$$

Outline

Multinomial coefficients

Divisions of n objects into r groups with size $n_1, \ldots, n_r$: We have

- n objects and r groups
- We want n_j objects in group j and $\sum_{j=1}^{r} n_j = n$

Notation: Set

$$\binom{n}{n_1, \ldots, n_r} = \frac{n!}{\prod_{j=1}^{r} (n_j!)}$$

Counting: We have

$$\# \text{ Divisions of } n \text{ objects into } r \text{ groups with size } n_1, \ldots, n_r = \binom{n}{n_1, \ldots, n_r}$$

Proof of counting

Number of choices for the ith group:

$$\binom{n - \sum_{j=1}^{i-1} n_j}{n_i}$$

Number of divisions: We have

$$\begin{aligned}
&\# \text{ Divisions of } n \text{ objects into } r \text{ groups with size } n_1, \ldots, n_r \\
&= \prod_{i=1}^{r} \binom{n - \sum_{j=1}^{i-1} n_j}{n_i} \\
&= \binom{n}{n_1, \ldots, n_r}
\end{aligned}$$

Example of multinomial coefficient (1)

Situation: Police department with 10 officers and

- 5 have to patrol the streets
- 2 are permanently working at the station
- 3 are on reserve at the station

Problem:
How many divisions do we get?

Example of multinomial coefficient (2)

Answer:

$$\frac{10!}{5!\,2!\,3!} = 2520$$

Tournament example

Situation: Tournament with $n = 2^m$ players
$\hookrightarrow$ How many outcomes?

Particular case:
Take $m = 3$, thus $n = 8$

Number of rounds: 3

Tournament example (2)

Counting number of outcomes for the first round:

$$\overbrace{\binom{8}{2,2,2,2}}^{\#\text{ pairings with order}} \quad \overbrace{\frac{1}{4!}}^{\text{No ordering}} \quad \overbrace{2^4}^{\text{Possible outcomes}} = \frac{8!}{4!}$$

Counting number of outcomes for second and third round:

$$\frac{4!}{2!} \quad \text{and} \quad \frac{2!}{1!}$$

Conclusion:

$$\frac{8!}{4!}\frac{4!}{2!}\frac{2!}{1!} = 8! = 40,320 \text{ possible outcomes}$$

Multinomial theorem

Theorem 5.

Let

- $x_1, x_2, \ldots, x_r \in \mathbb{R}$
- $n \geq 1$

Then

$$(x_1 + x_2 + \ldots + x_r)^n = \sum_{(n_1,\ldots,n_r) \in A_{n,r}} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

where

$$A_{n,r} = \{(n_1, \ldots, n_r) \in \mathbb{N}^r;\ n_1 + n_2 + \cdots + n_r = n\}$$