Conditional probability and independence

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Outline

Outline

Global objective

Aim: Introduce conditional probability, whose interest is twofold

1. Quantify the effect of a prior information on probabilities
2. If no prior information is available, then independence
   $\hookrightarrow$ simplification in probability computations

Outline

Example of conditioning

Dice tossing: We consider the following situation

- We throw 2 dice
- We look for $\mathbf{P}$(sum of 2 faces is 9)

Without prior information:

$$\mathbf{P}\,(\text{sum of 2 faces is 9}) = \frac{1}{9}$$

With additional information: If first face is $= 4$. Then

- Only 6 possible results: $(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)$
- Among them, only $(4,5)$ give sum$=9$
- Probability of having sum$=9$ becomes $\frac{1}{6}$

General definition

Definition 1.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events, such that $\mathbf{P}(F) > 0$

Then

$$\mathbf{P}\left(E \middle| F\right) = \frac{\mathbf{P}(E\,F)}{\mathbf{P}(F)}$$

Example: examination (1)

Situation:
Student taking a one hour exam

Hypothesis: For $x \in [0, 1]$ we have

$$\mathbf{P}(L_x) = \frac{x}{2}, \tag{1}$$

where the event L_x is defined by

$$L_x = \{\text{student finishes the exam in less than } x \text{ hour}\}$$

Question: Given that the student is still working after .75h
$\hookrightarrow$ Find probability that the full hour is used

Example: examination (2)

Model: We wish to find

$$\mathbf{P}\left(L_1^c \mid L_{.75}^c\right)$$

Computation: We have

$$\mathbf{P}\left(L_1^c \mid L_{.75}^c\right) = \frac{\mathbf{P}\left(L_1^c L_{.75}^c\right)}{\mathbf{P}\left(L_{.75}^c\right)} = \frac{\mathbf{P}\left(L_1^c\right)}{\mathbf{P}\left(L_{.75}^c\right)} = \frac{1-\mathbf{P}\left(L_1\right)}{1-\mathbf{P}\left(L_{.75}\right)}$$

Conclusion: Applying (1) we get

$$\mathbf{P}\left(L_1^c \mid L_{.75}^c\right) = .8$$

Simplification for uniform probabilities

General situation: We assume

- $S = \{s_1, \ldots, s_N\}$ finite.
- $\mathbf{P}(\{s_i\}) = \frac{1}{N}$ for all $1 \leq i \leq N$

Alert:
This is an important but very particular case of probability space

Conditional probabilities in this case:
Reduced sample space, i.e

Conditional on F, all outcomes in F are equally likely

Example: family distribution (1)

Situation:

The Popescu family has 10 kids

Questions:

1. If we know that 9 kids are girls
   $\hookrightarrow$ find the probability that all 10 kids are girls
2. If we know that the first 9 kids are girls
   $\hookrightarrow$ find the probability that all 10 kids are girls

Example: family distribution (2)

Model:

- $S = \{G, B\}^{10}$
- Uniform probability: for all $s \in S$,

$$\mathbf{P}(\{s\}) = \frac{1}{2^{10}}$$

Example: family distribution (3)

First conditioning: We take

$$F_1 = \{(G, \ldots, G); (G, \ldots, G, B); (G, \ldots, G, B, G); \cdots ; (B, G, \ldots, G)\}$$

Reduced sample space:
Each outcome in F_1 has probability $\frac{1}{11}$

Conditional probability:

$$\mathbf{P}\left(\{(G, \ldots, G)\} \,|\, F_1\right) = \frac{1}{11}$$

Example: family distribution (4)

Second conditioning: We take

$$F_2 = \{(G, \ldots, G); (G, \ldots, G, B)\}$$

Reduced sample space:
Each outcome in F_2 has probability $\frac{1}{2}$

Conditional probability:

$$\mathbf{P}\left(\{(G, \ldots, G)\} \middle| F_2\right) = \frac{1}{2}$$

Example: bridge game (1)

Bridge game:

- 4 players, E, W, N, S
- 52 cards dealt out equally to players

Conditioning: We condition on the set

$$F = \{N + S \text{ have a total of 8 spades}\}$$

Question: Conditioned on F,
Probability that E has 3 of the remaining 5 spades

Example: bridge game (2)

Model: We take

$$S = \{\text{Divisions of 52 cards in 4 groups}\}$$

and we have

- Uniform probability on S
- $|S| = \binom{52}{13,13,13,13} \simeq 5.36\ 10^{28}$

Reduced sample space: Conditioned on F,

$$\tilde{S} = \{\text{Combinations of 13 cards among 26 cards with 5 spades}\}$$

Example: bridge game (3)

Conditional probability:

$$\mathbf{P}\left(\text{E has 3 of the remaining 5 spades} \middle| F\right) = \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}} \simeq .339$$

Intersection and conditioning

Situation:

- Urn with 8 Red and 4 White balls
- Draw 2 balls without replacement

Question: Let

- R_1 = 1st ball drawn is red
- R_2 = 2nd ball drawn is red

Then find $\mathbf{P}(R_1 R_2)$

Intersection and conditioning (2)

Recall:

- Urn with 8 Red and 4 White balls
- Draw 2 balls without replacement

Computation: We have

$$\mathbf{P}(R_1 R_2) = \mathbf{P}(R_1)\mathbf{P}(R_2 | R_1)$$

Thus

$$\mathbf{P}(R_1 R_2) = \frac{8}{12}\frac{7}{11} = \frac{14}{33} \simeq .42$$

The multiplication rule

Proposition 2.

Let

- $\mathbf{P}$ a probability on a sample space S
- $E_1, \ldots, E_n$ n events

Then

$$\mathbf{P}\left(E_1 \cdots E_n\right) = \mathbf{P}\left(E_1\right) \prod_{k=1}^{n-1} \mathbf{P}\left(E_{k+1} \middle| E_1 \cdots E_k\right) \tag{2}$$

Proof

Expression for the rhs of (2):

$$\mathbf{P}(E_1)\ \frac{\mathbf{P}(E_1E_2)}{\mathbf{P}(E_1)}\ \frac{\mathbf{P}(E_1E_2E_3)}{\mathbf{P}(E_1E_2)} \cdots \frac{\mathbf{P}(E_1\cdots E_{n-1}E_n)}{\mathbf{P}(E_1\cdots E_{n-1})}$$

Conclusion:
By telescopic simplification

Example: deck of cards (1)

Situation:

- Ordinary deck of 52 cards
- Division into 4 piles of 13 cards

Question: If

$$E = \{\text{each pile has one ace}\},$$

compute $\mathbf{P}(E)$

Example: deck of cards (2)

Model: Set

$$
\begin{aligned}
E_1 &= \{\text{the ace of S is in any one of the piles}\} \\
E_2 &= \{\text{the ace of S and the ace of H are in different piles}\} \\
E_3 &= \{\text{the aces of S, H \& D are all in different piles}\} \\
E_4 &= \{\text{all 4 aces are in different piles}\}
\end{aligned}
$$

We wish to compute

$$\mathbf{P}(E_1 E_2 E_3 E_4)$$

Example: deck of cards (3)

Applying the multiplication rule: write

$$\mathbf{P}(E_1 E_2 E_3 E_4) = \mathbf{P}(E_1)\, \mathbf{P}(E_2 | E_1)\, \mathbf{P}(E_3 | E_1 E_2)\, \mathbf{P}(E_4 | E_1 E_2 E_3)$$

Computation of $\mathbf{P}(E_1)$: Trivially

$$\mathbf{P}(E_1) = 1$$

Computation of $\mathbf{P}(E_2 | E_1)$: Given E_1,

- Reduced space is
  $\{$51 labels given to all cards except for ace S$\}$
- $\mathbf{P}(E_2 | E_1) = \frac{51-12}{51} = \frac{39}{51}$

Example: deck of cards (4)

Other conditioned probabilities:

$$\mathbf{P}(E_3 | E_1 E_2) = \frac{50-24}{50} = \frac{26}{50},$$
$$\mathbf{P}(E_4 | E_1 E_2 E_3) = \frac{49-36}{49} = \frac{13}{49}$$

Conclusion: We get

$$\mathbf{P}(E) = \mathbf{P}(E_1)\,\mathbf{P}(E_2 | E_1)\,\mathbf{P}(E_3 | E_1 E_2)\,\mathbf{P}(E_4 | E_1 E_2 E_3)$$
$$= \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} \simeq .105$$

Outline

Thomas Bayes

Some facts about Bayes:

- England, 1701-1760
- Presbyterian minister
- Philosopher and statistician
- Wrote 2 books in entire life
- Bayes formula unpublished

Decomposition of $\mathbf{P}(E)$

Proposition 3.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events with $0 < \mathbf{P}(F) < 1$

Then

$$\mathbf{P}(E) = \mathbf{P}(E|\, F)\, \mathbf{P}(F) + \mathbf{P}(E|\, F^c)\, \mathbf{P}(F^c)$$

Bayes's formula

Proposition 4.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events with $0 < \mathbf{P}(F) < 1$

Then

$$\mathbf{P}\left(F|\, E\right) = \frac{\mathbf{P}\left(E|\, F\right)\mathbf{P}(F)}{\mathbf{P}\left(E|\, F\right)\mathbf{P}(F) + \mathbf{P}\left(E|\, F^c\right)\mathbf{P}(F^c)}$$

Example: insurance company (1)

Situation:

- Two classes of people:
  those who are accident prone and those who are not.
- Accident prone: probability .4 of accident in a one-year period
- Not accident prone: probab .2 of accident in a one-year period
- 30% of population is accident prone

Question:

Probability that a new policyholder will have an accident within a year of purchasing a policy?

Example: insurance company (2)

Model: Define

- A_1 = Policy holder has an accident in 1 year
- A = Accident prone

Then

- $S = \{(A_1, A); (A_1^c, A); (A_1, A^c); (A_1^c, A^c)\}$
- Probability: given indirectly by conditioning

Aim:
Compute $\mathbf{P}(A_1)$

Example: insurance company (3)

Given data:

$$\mathbf{P}(A_1 | A) = .4, \qquad \mathbf{P}(A_1 | A^c) = .2, \qquad \mathbf{P}(A) = .3$$

Application of Proposition 3:

$$\mathbf{P}(A_1) = \mathbf{P}(A_1 | A)\, \mathbf{P}(A) + \mathbf{P}(A_1 | A^c)\, \mathbf{P}(A^c)$$

We get

$$\mathbf{P}(A_1) = 0.4 \times 0.3 + 0.2 \times 0.7 = 26\%$$

Example: swine flu (1)

Situation:

We assume that 20% of a pork population has swine flu.
A test made by a lab gives the following results:

- Among 50 tested porks with flu, 2 are not detected
- Among 30 tested porks without flu, 1 is declared sick

Question:

Probability that a pork is healthy while his test is positive?

Example: swine flu (2)

Model: We set F = "Flu", T = "Positive test"
We have

$$\mathbf{P}(F) = \frac{1}{5}, \quad \mathbf{P}(T^c \mid F) = \frac{1}{25}, \quad \mathbf{P}(T \mid F^c) = \frac{1}{30}$$

Aim:
Compute $\mathbf{P}(F^c \mid T)$

Example: swine flu (3)

Application of Proposition 4:

$$\begin{aligned}
\mathbf{P}(F^c \mid T) &= \frac{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c)}{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c) + \mathbf{P}(T \mid F)\,\mathbf{P}(F)} \\
&= \frac{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c)}{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c) + [1 - \mathbf{P}(T^c \mid F)]\,\mathbf{P}(F)} \\
&= 0.12
\end{aligned}$$

Conclusion:
12% chance of killing swines without proper justification

Henri Poincaré

Some facts about Poincaré:

- Born in **Nancy**, 1854-1912
- Cousin of Raymond Poincaré
  $\hookrightarrow$ French president during WW1
- Mathematician and engineer
- Numerous contributions in
  - Celestial mechanics
  - Relativity
  - Gravitational waves
  - Topology
  - Differential equation

An example by Poincaré (1)

Situation:

- We are on a train
- Someone gets on the train and proposes to play a card game
- The unknown person wins

Question:
Probability that this person has cheated?

An example by Poincaré (2)

Model: We set

- p = probability to win without cheating
- q = probability that the unknown person has cheated
- W = "The unknown person wins"
- C = "The unknown person has cheated"

Hypothesis on probabilities: We assume

$$\mathbf{P}(W \mid C^c) = p, \quad \mathbf{P}(W \mid C) = 1, \quad \mathbf{P}(C) = q$$

Aim:
Compute $\mathbf{P}(C \mid W)$

Applications (4)

Application of Proposition 4:

$$\mathbf{P}(C \mid W) = \frac{\mathbf{P}(W \mid C)\,\mathbf{P}(C)}{\mathbf{P}(W \mid C)\,\mathbf{P}(C) + \mathbf{P}(W \mid C^c)\,\mathbf{P}(C^c)} = \frac{q}{q + p(1-q)}$$

Remarks:

(1) We have $\mathbf{P}(C \mid W) \geq q = \mathbf{P}(C)$.
$\hookrightarrow$ the unknown's win increases his probability to cheat

(2) We have

$$\lim_{p \to 0} \mathbf{P}(C \mid W) = 1$$

Odds

Definition 5.

Let

- $\mathbf{P}$ a probability on a sample space S
- A an event

We define the odds of A by

$$\frac{\mathbf{P}(A)}{\mathbf{P}(A^c)} = \frac{\mathbf{P}(A)}{1 - \mathbf{P}(A)}$$

Odds and conditioning

Proposition 6.

Situation: We have

- An hypothesis H, true with probability $\mathbf{P}(H)$
- A new evidence E

Formula: The odds of H after evidence E are given by

$$\frac{\mathbf{P}(H|\,E)}{\mathbf{P}(H^c|\,E)} = \frac{\mathbf{P}(H)}{\mathbf{P}(H^c)}\,\frac{\mathbf{P}(E|\,H)}{\mathbf{P}(E|\,H^c)}$$

Proof

Inversion of conditioning: We have

$$\mathbf{P}(H|\,E) = \frac{\mathbf{P}(E|\,H)\,\mathbf{P}(H)}{\mathbf{P}(E)}$$

$$\mathbf{P}(H^c|\,E) = \frac{\mathbf{P}(E|\,H^c)\,\mathbf{P}(H^c)}{\mathbf{P}(E)}$$

Conclusion:

$$\frac{\mathbf{P}(H|\,E)}{\mathbf{P}(H^c|\,E)} = \frac{\mathbf{P}(H)}{\mathbf{P}(H^c)}\,\frac{\mathbf{P}(E|\,H)}{\mathbf{P}(E|\,H^c)}$$

Example: coin tossing (1)

Situation:

- Urn contains two type A coins and one type B coin.
- When a type A coin is flipped, it comes up heads with probability $\frac{1}{4}$
- When a type B coin is flipped, it comes up heads with probability $\frac{3}{4}$
- A coin is randomly chosen from the urn and flipped

Question:

Given that the flip landed on heads

$\hookrightarrow$ What is the probability that it was a type A coin?

Example: coin tossing (2)

Model: We set

- A = type A coin flipped
- B = type B coin flipped
- H = Head obtained

Data:

$$\mathbf{P}(A) = \frac{2}{3}, \qquad \mathbf{P}(H|\, A) = \frac{1}{4}, \qquad \mathbf{P}(H|\, B) = \frac{3}{4}$$

Aim:
Compute $\mathbf{P}(A|\, H)$

Example: coin tossing (3)

Application of Proposition 6:

$$\frac{\mathbf{P}(A|\, H)}{\mathbf{P}(B|\, H)} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)} \, \frac{\mathbf{P}(H|\, A)}{\mathbf{P}(H|\, B)}$$

Numerical result: We get

$$\frac{\mathbf{P}(A|\, H)}{\mathbf{P}(B|\, H)} = \frac{2/3}{1/3} \, \frac{1/4}{3/4} = \frac{2}{3}$$

Therefore

$$\mathbf{P}(A|\, H) = \frac{2}{5}$$

Generalization of Proposition 3

Proposition 7.

Let

- $\mathbf{P}$ a probability on a sample space S
- $F_1, \ldots, F_n$ partition of S, i.e
  - F_i mutually exclusive
  - $\cup_{i=1}^{n} F_i = S$
- E another event

Then we have

$$\mathbf{P}(E) = \sum_{i=1}^{n} \mathbf{P}(E | F_i)\, \mathbf{P}(F_i)$$

Generalization of Proposition 4

Proposition 8.

Let

- $\mathbf{P}$ a probability on a sample space S
- $F_1, \ldots, F_n$ partition of S, i.e
  - F_i mutually exclusive
  - $\cup_{i=1}^n F_i = S$
- E another event

Then we have

$$\mathbf{P}(F_j | E) = \frac{\mathbf{P}(E | F_j)\, \mathbf{P}(F_j)}{\sum_{i=1}^n \mathbf{P}(E | F_i)\, \mathbf{P}(F_i)}$$

Example: card game (1)

Situation:

- 3 cards identical in form (say Jack)
- Coloring of the cards on both faces:
  - 1 card RR
  - 1 card BB
  - 1 card RB
- 1 card is randomly selected, with upper side R

Question:

What is the probability that the other side is B?

Example: card game (2)

Model: We define the events

- RR: chosen car is all red
- BB: chosen card is all black
- RB: chosen card is red and black
- R: upturned side of chosen card is red

Aim:
Compute $\mathbf{P}(RB|\,R)$

Example: card game (3)

Application of Proposition 8:

$$\mathbf{P}(RB|\,R) = \frac{\mathbf{P}(R|\,RB)\,\mathbf{P}(RB)}{\mathbf{P}(R|\,RR)\,\mathbf{P}(RR) + \mathbf{P}(R|\,RB)\,\mathbf{P}(RB) + \mathbf{P}(R|\,BB)\,\mathbf{P}(BB)}$$

Numerical values:

$$\mathbf{P}(RB|\,R) = \frac{\frac{1}{2} \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3}} = \frac{1}{3}$$

Example: disposable flashlights

Situation:

- Bin containing 3 different types of disposable flashlights
- Proba that a type 1 flashlight will give over 100 hours of use is .7
- Corresponding probabilities for types 2 & 3: .4 and .3
- 20% of the flashlights are type 1, 30% are type 2, and 50% are type 3

Questions:

1. What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
2. Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, for $j = 1, 2, 3$?

Example: disposable flashlights (2)

Model: We define the events

- A: flashlight chosen gives more than 100h of use
- F_j: type j is chosen

Aim 1:
Compute $\mathbf{P}(A)$

Example: disposable flashlights (3)

Application of Proposition 7:

$$\mathbf{P}(A) = \sum_{j=1}^{3} \mathbf{P}(A|\, F_j)\ \mathbf{P}(F_j)$$

Numerical values:

$$\mathbf{P}(A) = 0.7 \times 0.2 + 0.4 \times 0.3 + 0.3 \times 0.5 = .41$$

Example: disposable flashlights (4)

Aim 2:
Compute $\mathbf{P}(F_1 | A)$

Application of Proposition 8:

$$\mathbf{P}(F_1 | A) = \frac{\mathbf{P}(A | F_1)\,\mathbf{P}(F_1)}{\mathbf{P}(A)}$$

Numerical value:

$$\mathbf{P}(F_1 | A) = \frac{0.7 \times 0.2}{0.41} = \frac{14}{41} \simeq 41\%$$

Outline

Definition of independence

Definition 9.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events

Then E and F are independent if

$$\mathbf{P}(E\,F) = \mathbf{P}(E)\,\mathbf{P}(F)$$

Notation:

E and F independent denoted by $E \perp\!\!\!\perp F$

Some remarks

Interpretation: If $E \perp\!\!\!\perp F$, then

$$\mathbf{P}\left(E \middle| F\right) = \mathbf{P}(E),$$

that is the knowledge of F does not affect $\mathbf{P}(E)$

Warning: Independent $\neq$ mutually exclusive!
Specifically

$$\begin{aligned} A, B \text{ mutually exclusive} \quad &\Rightarrow \quad \mathbf{P}(A\,B) = 0 \\ A, B \text{ independent} \quad &\Rightarrow \quad \mathbf{P}(A\,B) = \mathbf{P}(A)\,\mathbf{P}(B) \end{aligned}$$

Therefore A et B both independent and mutually exclusive
$\hookrightarrow$ we have either $\mathbf{P}(A) = 0$ or $\mathbf{P}(B) = 0$

Example: dice tossing (1)

Experiment: We throw two dice

Sample space:

- $S = \{1, \ldots, 6\}^2$
- $\mathbf{P}(\{(s_1, s_2)\}) = \frac{1}{36}$ for all $(s_1, s_2) \in S$

Events: We consider

$$A = \text{"1}^{\text{st}} \text{ outcome is 1"}, \qquad B = \text{"2}^{\text{nd}} \text{ outcome is 4"}$$

Question:
Do we have $A \perp\!\!\!\perp B$?

Example: dice tossing (2)

Description of A and B:

$$B = \{1\} \times \{1, \ldots, 6\}, \quad \text{and} \quad B = \{1, \ldots, 6\} \times \{4\}.$$

Probabilities for A and B: We have

$$\mathbf{P}(A) = \frac{|A|}{36} = \frac{1}{6}, \qquad \mathbf{P}(B) = \frac{|B|}{36} = \frac{1}{6}$$

Description of $A\,B$: We have $A\,B = \{(1, 4)\}$. Thus

$$\mathbf{P}(A\,B) = \frac{1}{36} = \mathbf{P}(A)\,\mathbf{P}(B)$$

Conclusion: A and B are independent

Example: tossing n coins (1)

Experiment:
Tossing a coin n times

Events: We consider

$$A = \text{"At most one Head"}$$
$$B = \text{"At least one Head and one Tail"}$$

Question:
Are there values of n such that $A \perp\!\!\!\perp B$?

Example: tossing n coins (2)

Model: We take

- $S = \{h, t\}^n$
- $\mathbf{P}(\{s\}) = \frac{1}{2^n}$ for all $s \in S$

Description of A and B:

$$\begin{aligned} A &= \{(t,\ldots,t),\,(h,t,\ldots,t),(t,h,t,\ldots,t),\,(t,\ldots,t,h)\} \\ B &= \{(h,\ldots,h),\,(t,\ldots,t)\}^c \end{aligned}$$

Example: tossing n coins (3)

Computing probabilities for A and B: We have

$$\begin{aligned} \mathbf{P}(A) &= \frac{|A|}{2^n} = \frac{n+1}{2^n} \\ \mathbf{P}(B) &= 1 - \mathbf{P}(B^c) = 1 - \frac{1}{2^{n-1}} \end{aligned}$$

Description of $A\,B$ and

$$A\,B = A \backslash \{(f, \ldots, f)\} \quad \Rightarrow \quad \mathbf{P}(A\,B) = \frac{n}{2^n}$$

Example: tossing n coins (4)

Checking independence: We have $A \perp\!\!\!\perp B$ iff

$$\frac{n+1}{2^n}\left(1-\frac{1}{2^{n-1}}\right)=\frac{n}{2^n} \quad \Longleftrightarrow \quad n-2^{n-1}+1=0$$

Conclusion: One can check that

$$x \mapsto x-2^{x-1}+1$$

vanishes for $x=3$ only on $\mathbb{R}_+$. Thus

We have $A \perp\!\!\!\perp B$ iff $n=3$

Independence and complements

Proposition 10.

Let

- **P** a probability on a sample space S
- E, F two events
- We assume that $E \perp\!\!\!\perp F$

Then

$$E \perp\!\!\!\perp F^c, \qquad E^c \perp\!\!\!\perp F, \qquad E^c \perp\!\!\!\perp F^c$$

Proof

Decomposition of $\mathbf{P}(E)$: Write

$$\begin{aligned}\mathbf{P}(E) &= \mathbf{P}(E\,F) + \mathbf{P}(E\,F^c) \\ &= \mathbf{P}(E)\,\mathbf{P}(F) + \mathbf{P}(E\,F^c)\end{aligned}$$

Expression for $\mathbf{P}(E\,F^c)$: From the previous expression we have

$$\begin{aligned}\mathbf{P}(E\,F^c) &= \mathbf{P}(E) - \mathbf{P}(E)\,\mathbf{P}(F) \\ &= \mathbf{P}(E)\,(1 - \mathbf{P}(F)) \\ &= \mathbf{P}(E)\mathbf{P}(F^c)\end{aligned}$$

Conclusion:
$E \perp\!\!\!\perp F^c$

Counterexample: independence of 3 events (1)

Warning:
In certain situations we have A, B, C pairwise independent, however

$$\mathbf{P}(A \cap B \cap C) \neq \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C)$$

Example: tossing two dice

- $S = \{1, \ldots, 6\}^2$
- $\mathbf{P}(\{(s_1, s_2)\}) = \frac{1}{36}$ for all $(s_1, s_2) \in S$

Events: Define

$$A = \text{"even number for the } 1^{\text{st}} \text{ outcome"}$$
$$B = \text{"odd number for the } 2^{\text{nd}} \text{ outcome"}$$
$$C = \text{"same parity for the two outcomes"}$$

Counterexample: independence of 3 events (2)

Description of A, B, C:

$$\begin{aligned} A &= \{2,4,6\} \times \{1,\ldots,6\} \\ B &= \{1,\ldots,6\} \times \{1,3,5\} \\ C &= (\{2,4,6\} \times \{2,4,6\}) \cup (\{1,3,5\} \times \{1,3,5\}) \end{aligned}$$

Pairwise independence: we find

$$A \perp\!\!\!\perp B, \; A \perp\!\!\!\perp C \text{ and } B \perp\!\!\!\perp C$$

Independence of the 3 events: We have $A \cap B \cap C = \emptyset$. Thus

$$0 = \mathbf{P}(A \cap B \cap C) \neq \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C) = \frac{1}{8}$$

Independence of 3 events

Definition 11.

Let

- $\mathbf{P}$ a probability on a sample space S
- 3 events A_1, A_2, A_3

We say that A_1, A_2, A_3 are independent if

$$\begin{aligned} \mathbf{P}(A_1 A_2) &= \mathbf{P}(A_1)\,\mathbf{P}(A_2), \quad \mathbf{P}(A_1 A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_3) \\ \mathbf{P}(A_2 A_3) &= \mathbf{P}(A_2)\,\mathbf{P}(A_3) \end{aligned}$$

and

$$\mathbf{P}(A_1 A_2 A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3)$$

Independence of n events

Definition 12.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $A_1, A_2, \ldots, A_n$

We say that $A_1, A_2, \ldots, A_n$ are independent if
for all $2 \le r \le n$ and $j_1 < \cdots < j_r$ we have

$$\mathbf{P}(A_{j_1} A_{j_2} \cdots A_{j_r}) = \mathbf{P}(A_{j_1}) \, \mathbf{P}(A_{j_2}) \cdots \mathbf{P}(A_{j_r})$$

Independence of an ∞ number of events

Definition 13.

Let

- $\mathbf{P}$ a probability on a sample space S
- A sequence of events $\{A_i;\ i \geq 1\}$

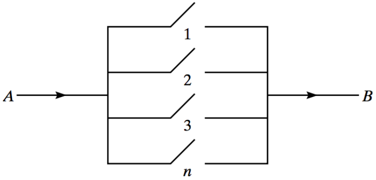
We say that the A_i's are independent if
for all $2 \leq r < \infty$ and $j_1 < \cdots < j_r$ we have

$$\mathbf{P}(A_{j_1} A_{j_2} \cdots A_{j_r}) = \mathbf{P}(A_{j_1})\, \mathbf{P}(A_{j_2}) \cdots \mathbf{P}(A_{j_r})$$

Example: parallel system (1)

Situation:

- Parallel system with n components
- All components are independent
- Probability that i-th component works: p_i



Question:

Probability that the system functions

Example: parallel system (2)

Model: We take

- $S = \{0,1\}^n$
- Probability $\mathbf{P}$ on S defined by

$$\mathbf{P}(\{(s_1,\ldots,s_n)\}) = \prod_{i=1}^{n} p_i^{s_i}(1-p_i)^{1-s_i}$$

Events:

A = "System functions" , A_i = "i-th component functions"

Facts about A_i's:
The events A_i are independent and $\mathbf{P}(A_i) = p_i$

Example: parallel system (3)

Computations for $\mathbf{P}(A^c)$:

$$\begin{aligned} \mathbf{P}(A^c) &= \mathbf{P}\left(\cap_{i=1}^n A_i^c\right) \\ &= \prod_{i=1}^n \mathbf{P}\left(A_i^c\right) \\ &= \prod_{i=1}^n (1 - p_i) \end{aligned}$$

Conclusion:

$$\mathbf{P}(A) = 1 - \prod_{i=1}^n (1 - p_i)$$

Example: rolling dice (1)

Experiment:

- Roll a pair of dice
- Outcome: sum of faces

Events:

- $E_n =$ "no 5 or 7 on first $n-1$ trials, then 5 on n-th trial"
- $E = \cup_{n\geq 1} E_n =$ "5 appears before 7"

Question:

Compute $\mathbf{P}(E)$

Example: rolling dice (2)

Computation for $\mathbf{P}(E_n)$: by independence

$$\mathbf{P}\left(E_n\right) = \left(1 - \frac{10}{36}\right)^{n-1} \frac{4}{36} = \left(\frac{13}{36}\right)^{n-1} \frac{1}{9}$$

Computation for $\mathbf{P}(E)$:

$$\mathbf{P}(E) = \sum_{n=1}^{\infty} \mathbf{P}\left(E_n\right) = \frac{1}{9} \frac{1}{1 - \frac{13}{36}}$$

Thus

$$\mathbf{P}(E) = \frac{2}{5}$$

Same example with conditioning (1)

New events: We set

- E = "5 appears before 7"
- F_5 = "1st trial gives 5"
- F_7 = "1st trial gives 7"
- H = "1st trial gives an outcome $\neq$ 5,7"

Same example with conditioning (2)

Conditional probabilities:

$$\mathbf{P}(E|\, F_5) = 1, \quad \mathbf{P}(E|\, F_7) = 0, \quad \mathbf{P}(E|\, H) = \mathbf{P}(E)$$

Justification: $E \perp\!\!\!\perp H$ since

$$E\,H = H \cap \{\text{Event which depends on } i\text{-th trials with } i \geq 2\}$$

Same example with conditioning (3)

Applying Proposition 7:

$$\mathbf{P}(E) = \mathbf{P}(E|\, F_5)\; \mathbf{P}(F_5) + \mathbf{P}(E|\, F_7)\; \mathbf{P}(F_7) + \mathbf{P}(E|\, H)\; \mathbf{P}(H) \quad (3)$$

Computation: We get

$$\mathbf{P}(E) = \frac{1}{9} + \frac{13}{18}\mathbf{P}(E),$$

and thus

$$\mathbf{P}(E) = \frac{2}{5}$$

Problem of the points

Experiment:

- Independent trials
- For each trial, success with probability p

Question:

What is the probability that n successes occur before m failures?

Pascal's solution

Notation: set

$$A_{n,m} = \text{"}n \text{ successes occur before } m \text{ failures"}, \qquad P_{n,m} = \mathbf{P}(A_{n,m})$$

Conditioning on 1st trial: Like in (3) we get

$$P_{n,m} = pP_{n-1,m} + (1-p)P_{n,m-1} \tag{4}$$

Initial conditions:

$$P_{n,0} = p^n, \qquad P_{0,m} = (1-p)^m \tag{5}$$

Strategy:
Solve difference equation (4) with initial condition (5)

Fermat's solution

Expression for $A_{n,m}$: Write

$$A_{n,m} = \text{"at least } n \text{ successes in } m+n-1 \text{ trials"}$$

Thus $A_{n,m} = \cup_{k=n}^{m+n-1} E_{k,m,n}$ with

$$E_{k,m,n} = \text{"exactly } k \text{ successes in } m+n-1 \text{ trials"}$$

Expression for $P_{n,m}$: We get

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$

Outline

$\mathbf{P}(\cdot|\, F)$ is a probability

Proposition 14.

Let

- $\mathbf{P}$ a probability on a sample space S
- F an event such that $\mathbf{P}(F) > 0$

Then

$$\mathbf{Q} : E \mapsto \mathbf{P}(E|\, F)$$

is a probability

Proof (1)

$0 \leq \mathbf{Q}(E) \leq 1$:

$$0 \leq \mathbf{Q}(E) = \frac{\mathbf{P}(E\,F)}{\mathbf{P}(F)} \leq \frac{\mathbf{P}(F)}{\mathbf{P}(F)} = 1$$

$\mathbf{Q}(S) = 1$:

$$\mathbf{Q}(S) \quad = \quad \frac{\mathbf{P}(S\,F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(F)}{\mathbf{P}(F)} = 1$$

Proof (2)

Additivity: Let $\{E_n;\, n \geq 1\}$ be a family of mutually exclusive events. We claim that

$$\mathbf{Q}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbf{Q}(E_n)$$

Justification:

$$\mathbf{Q}\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\mathbf{P}\left(\left(\bigcup_{n=1}^{\infty} E_n\right) F\right)}{\mathbf{P}(F)}$$

$$= \frac{\mathbf{P}\left(\bigcup_{n=1}^{\infty} (E_n F)\right)}{\mathbf{P}(F)} = \frac{\sum_{n=1}^{\infty} \mathbf{P}(E_n F)}{\mathbf{P}(F)} = \sum_{n=1}^{\infty} \mathbf{Q}(E_n)$$

Intersection and conditioning – Part 2

Proposition 15.

Let

- $\mathbf{P}$ a probability on a sample space S
- E_1, E_2 two events
- F an event such that $\mathbf{P}(F) > 0$

Then

$$\mathbf{P}(E_1 | F) = \mathbf{P}(E_1 | E_2 F)\, \mathbf{P}(E_2 | F) + \mathbf{P}(E_1 | E_2^c F)\, \mathbf{P}(E_2^c | F)$$

Proof

Strategy:
Apply Proposition 3 to the probability $\mathbf{Q}$ of Proposition 14

$$\mathbf{Q}(E_1) = \mathbf{Q}(E_1 | E_2)\,\mathbf{Q}(E_2) + \mathbf{Q}(E_1 | E_2^c)\,\mathbf{Q}(E_2^c)$$

Computing the conditional probabilities:

$$\mathbf{Q}(E_1 | E_2) = \mathbf{P}(E_1 | E_2 F), \qquad \mathbf{Q}(E_1 | E_2^c) = \mathbf{P}(E_1 | E_2^c F)$$

Conclusion:

$$\mathbf{P}(E_1 | F) = \mathbf{P}(E_1 | E_2 F)\ \mathbf{P}(E_2 | F) + \mathbf{P}(E_1 | E_2^c F)\ \mathbf{P}(E_2^c | F)$$

Example: insurance company – Part 2 (1)

Situation:

- Two classes of people:
  those who are accident prone and those who are not.
- Accident prone: probability .4 of accident in a one-year period
- Not accident prone: probab .2 of accident in a one-year period
- 30% of population is accident prone

Question:

Probability that a new policyholder will have an accident within her/his second year of purchasing a policy if we know she/he had an accident in his first year?

Example: insurance company (2)

Model: Define

- A_1 = Policy holder has an accident in his first year
- A_2 = Policy holder has an accident in his second year
- A = Accident prone

Given data:

$$\mathbf{P}(A_1 | A) = .4, \qquad \mathbf{P}(A_1 | A^c) = .2, \qquad \mathbf{P}(A) = .3$$

Aim:
Compute $\mathbf{P}(A_2 | A_1)$

Example: insurance company (3)

Application of Proposition 15:

$$\mathbf{P}(A_2|\, A_1) = \mathbf{P}(A_2|\, A\, A_1)\ \mathbf{P}(A|\, A_1) + \mathbf{P}(A_2|\, A^c\, A_1)\ \mathbf{P}(A^c|\, A_1)$$

Computation of conditional probabilities:

$$\mathbf{P}(A_2|\, A\, A_1) = .4, \qquad \mathbf{P}(A_2|\, A^c\, A_1) = .2$$

Example: insurance company (4)

Computation of conditional probabilities (2):

$$\mathbf{P}(A|\, A_1) = \frac{\mathbf{P}(A_1|\, A)\ \mathbf{P}(A)}{\mathbf{P}(A_1)} = \frac{0.4 \times 0.3}{0.26} = \frac{6}{13}$$

and

$$\mathbf{P}(A^c|\, A_1) = 1 - \mathbf{P}(A|\, A_1) = \frac{7}{13}$$

Conclusion:

$$\mathbf{P}(A_2|\, A_1) = 0.4 \times \frac{6}{13} + 0.2 \times \frac{7}{13} \simeq 29\%$$

Matching problem (1)

Situation:

- n men take off their hats
- Hats are mixed up
- Then each man selects his hat at random
- Match: if a man selects his own hat

Questions:

1. Probability of no match
2. Probability of exactly k matches

Matching problem (2)

Model: We set

- E = no match
- M = first man selects his hat
- $P_n = \mathbf{P}(E)$

Conditioning on M:

$$
\begin{aligned}
P_n &= \mathbf{P}(E|\,M)\,\mathbf{P}(M) + \mathbf{P}(E|\,M^c)\,\mathbf{P}(M^c) \\
&= \mathbf{P}(E|\,M^c)\,\frac{n-1}{n}
\end{aligned}
$$

Matching problem (3)

New situation on M^c:

- $n-1$ hats with $n-1$ men
- 1 extra man with no hat
- 1 extra hat with no man
- Set $N =$ "extra man selects extra hat"

Conditioning on N:

$$\mathbf{P}\left(E \mid M^c\right) = \mathbf{P}\left(E\, N \mid M^c\right) + \mathbf{P}\left(E \mid N^c\, M^c\right)\, \mathbf{P}\left(N^c \mid M^c\right) \qquad (6)$$

Matching problem (4)

Recall:

$$\mathbf{P}\left(E \mid M^c\right) = \mathbf{P}\left(E\, N \mid M^c\right) + \mathbf{P}\left(E \mid N^c\, M^c\right)\, \mathbf{P}\left(N^c \mid M^c\right) \tag{7}$$

New situation if N^c occurs: since extra man does not select extra hat

- Declare extra hat as extra man's
- Whole situation equivalent to $(n-1)$ mixed hats

New situation if N occurs:

- 1 extra man selects extra hat
- We are left with $(n-2)$ mixed hats

Consequence on (7):

$$\mathbf{P}\left(E \mid M^c\right) = P_{n-1} + \frac{1}{n-1} P_{n-2} \tag{8}$$

Matching problem (5)

Putting together (6) and (8): We get

$$P_n = \frac{n-1}{n} P_{n-1} + \frac{1}{n} P_{n-2} \quad \Longleftrightarrow \quad P_n - P_{n-1} = -\frac{1}{n} (P_{n-1} - P_{n-2})$$

Initial data:

$$P_1 = 0, \qquad P_2 = \frac{1}{2}$$

Solution of difference equation:

$$P_n = \sum_{j=2}^{n} \frac{(-1)^j}{j!}$$

Matching problem (6)

Events for the k-match problem: We set

- E_k = exactly k matches
- F_j = match for man j

Successive conditioning: For $1 \leq j_1 < \cdots < j_k \leq n$ we get

$$\begin{aligned} \mathbf{P}\left(F_{j_1} \cdots F_{j_k} E_k\right) &= \frac{1}{n} \frac{1}{n-1} \cdots \frac{1}{n-(k-1)} P_{n-k} \\ &= \frac{(n-k)!}{n!} P_{n-k} \end{aligned}$$

Matching problem (7)

Recall:

$$\mathbf{P}\left(F_{j_1}\cdots F_{j_k}\,E_k\right)=\frac{(n-k)!}{n!}\,P_{n-k}$$

Computing $\mathbf{P}(E_k)$: We have

$$\begin{aligned}\mathbf{P}\left(E_k\right) &= \sum_{1\le j_1<\cdots<j_k\le n}\mathbf{P}\left(F_{j_1}\cdots F_{j_k}\,E_k\right)\\ &= \binom{n}{k}\frac{(n-k)!}{n!}\,P_{n-k}\end{aligned}$$

Therefore

$$\mathbf{P}\left(E_k\right)=\frac{1}{k!}\,P_{n-k}$$

Conditional independence

Definition 16.

Let

- $\mathbf{P}$ a probability on a sample space S
- E_1, E_2 two events
- F an event such that $\mathbf{P}(F) > 0$

We say that E_1, E_2 are independent conditionally on F if

$$\mathbf{P}(E_1 E_2 | F) = \mathbf{P}(E_1 | F) \, \mathbf{P}(E_2 | F)$$

Laplace's rule of succession (1)

Experiment:

- $k + 1$ coins in a box
- Probability of Heads for i-th coin: $\frac{i}{k}$, $i = 0, \ldots, k$
- Coin randomly selected
- Observation: n successive Heads

Question:

Probability that the $(n + 1)$-th flip is also Head

Laplace's rule of succession (2)

Model: We set

- $C_i = i$-th coin initially selected
- $F_n =$ first n flips result in heads
- $H =$ (n+1)-th flip is a head

Aim:
Find $\mathbf{P}(H | F_n)$

Laplace's rule of succession (3)

Application of Proposition 15:

$$\mathbf{P}\left(H \mid F_n\right) = \sum_{i=0}^{k} \mathbf{P}\left(H \mid C_i F_n\right) \mathbf{P}\left(C_i \mid F_n\right)$$

Hypothesis:
The flips are independent conditionally on C_i

Consequence:

$$\mathbf{P}\left(H \mid C_i F_n\right) = \mathbf{P}\left(H \mid C_i\right) = \frac{i}{k}$$

Laplace's rule of succession (4)

Application of Proposition 8:

$$\mathbf{P}\left(C_i \middle| F_n\right) = \frac{\mathbf{P}\left(F_n \middle| C_i\right) \mathbf{P}\left(C_i\right)}{\sum_{j=0}^{k} \mathbf{P}\left(F_n \middle| C_j\right) \mathbf{P}\left(C_j\right)}$$

Consequence of conditional independence:

$$\mathbf{P}\left(C_i \middle| F_n\right) = \frac{\left(\frac{i}{k}\right)^n \frac{1}{k+1}}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^n \frac{1}{k+1}}$$

Thus

$$\mathbf{P}\left(C_i \middle| F_n\right) = \frac{\left(\frac{i}{k}\right)^n}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^n}$$

Laplace's rule of succession (5)

Conclusion:

$$\mathbf{P}\left(H \middle| F_n\right) = \frac{\sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+1}}{\sum_{j=0}^{k} \left(\frac{j}{k}\right)^{n}}$$

Approximation: For n large,

$$\mathbf{P}\left(H \middle| F_n\right) \simeq \frac{\int_0^1 x^{n+1}\, dx}{\int_0^1 x^n\, dx} = \frac{n+1}{n+2}$$