MA/STAT 519: Introduction to Probability
Fall 2018, Final Examination

Instructor: Yip

- This test booklet has FIVE QUESTIONS, totaling 100 points for the whole test. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**. One two-sided-8×11 formula sheet is allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Department:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
5.(20 pts)	
Total (100 pts)	

1. The pdf of a Cauchy random variable with (scale-)parameter $\alpha > 0$ is given by

$$p_\alpha(x) = \frac{\alpha}{\pi(x^2+\alpha^2)}, \quad -\infty < x < \infty.$$

You are given the fact that if X and Y are independent Cauchy random variables with parameters α and β, then $X+Y$ is a Cauchy random variable with parameter $\alpha+\beta$.

(a) Explain why Cauchy random variable is *infinitely divisible*, i.e. given a Cauchy random variable X with parameter α, for each positive integer n, there are *iid* random variables $Y_1, Y_2, \ldots, Y_n$ such that $Y_1 + Y_2 + \cdots + Y_n$ has the same distribution as X.

(b) Let $X_1, X_2, \ldots, X_n$ be iid Cauchy random variables with paramter 1. Find the pdf of

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

(a) $\text{Cauchy}(\alpha) + \text{Cauchy}(\beta) \overset{\mathcal{D}}{\sim} \text{Cauchy}(\alpha+\beta)$

Hence

$$\underbrace{\text{Cauchy}\left(\tfrac{\alpha}{n}\right) + \text{Cauchy}\left(\tfrac{\alpha}{n}\right) + \cdots + \text{Cauchy}\left(\tfrac{\alpha}{n}\right)}_{n \text{ ind. summands}}$$

$$= \text{Cauchy}\left(\tfrac{\alpha}{n} + \tfrac{\alpha}{n} + \cdots + \tfrac{\alpha}{n}\right)$$

$$= \text{Cauchy}(\alpha)$$

(b) $X_1 + X_2 + \cdots + X_n = Y \overset{\mathcal{D}}{\sim} \text{Cauchy}(1+1+\cdots+1)$

$$= \text{Cauchy}(n)$$

Hence pdf of Y is $p_Y(y) = \dfrac{n}{\pi(y^2+n^2)}$

You can use this blank page.

Let $Z = \frac{Y}{n}$

By change of variable formula, the pdf of Z is given by

$$p_Z(z) = p_Y(y)\left|\frac{dy}{dz}\right|$$

$$= p_Y(y)\, n$$

$$= p_Y(nz)\, n$$

$$= \frac{n}{\pi(n^2z^2+n^2)} \times n$$

$$= \frac{1}{\pi(z^2+1)}$$

Hence $\frac{X_1+X_2+\cdots+X_n}{n} = Z$

$\overset{\mathcal{D}}{\sim}$ Cauchy(1)

2. Suppose n balls are distributed in n boxes in such a way that each ball chooses a box independently of each other.

   (a) What is the probability that Box #1 is empty?

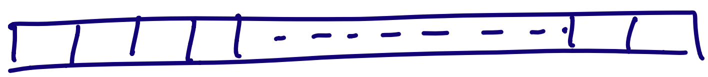
   (b) What is the probability that only Box #1 is empty?

   (c) What is the probability that only one box is empty?

   (d) Given that Box #1 is empty, what is the probability that only one box is empty?

   (e) Given that only one box is empty, what is the probability that Box #1 is empty?

n balls $\longrightarrow$ n boxes

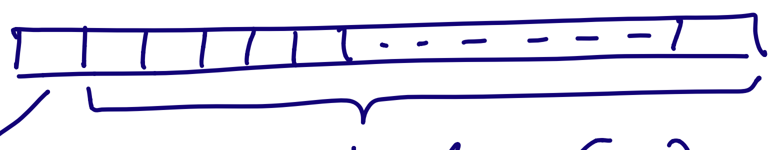


(a) $\text{Ans} = \dfrac{(n-1)^n}{n^n}$

$(n-1)^n$: no. of choices (in $n-1$ boxes)

n^n: total no. of choices

(b) n balls



0 balls

n balls in $(n-1)$-boxes

Hence all of the $n-1$ boxes each has one ball except one having two balls.

$\text{Ans} = \dfrac{\binom{n}{2}(n-1)!}{n^n}$

$\binom{n}{2}$: choose a group of 2 balls

$(n-1)!$: put into $(n-1)$ balls

n^n: total no. of ways

[Alternatively, make use of multi-nomial

You can use this blank page.

distribution:

The distribution of balls is given by:

$$x_1 + x_2 + \cdots + x_n = n,$$

with $x_1 = 0$, $x_2, x_3, \ldots, x_n \geq 1$.

Hence all of $x_2, x_3, \ldots, x_n = 1$, except one, which equals two.

Hence probability

$$= \sum \frac{n!}{x_1! \, x_2! \cdots x_n!} \left(\frac{1}{n}\right)^{x_1} \left(\frac{1}{n}\right)^{x_2} \cdots \left(\frac{1}{n}\right)^{x_n}$$

$$= (n-1) \times \frac{n!}{0! \, 1! \, 1! \cdots 2!} \frac{1}{n^n} = \frac{n(n-1)}{2} \frac{(n-1)!}{n^n}$$

choose one box with 2 balls

$$= \binom{n}{2} \frac{(n-1)!}{n^n}. \;]$$

(c): $\text{Ans}(c) = \underbrace{n \times \text{Ans}(b)} = n \binom{n}{2} \frac{(n-1)!}{n^n}$

any of the n balls can be empty.

You can use this blank page.

(d) $P\left(\text{only one box is empty} \mid \text{Box \#1 is empty}\right)$

$$= \frac{P(\text{only one box is empty} \cap \text{Box \# one is empty})}{P(\text{Box \#1 is empty})}$$

$$= \frac{P(\text{only Box \#1 is empty})}{P(\text{Box \#1 is empty})}$$

$$= \frac{(b)}{(a)} = \frac{\binom{n}{2}\frac{(n-1)!}{n^n}}{\frac{(n-1)^n}{n^n}} = \frac{\binom{n}{2}(n-1)!}{(n-1)^n}$$

(e) $\frac{1}{n}$ (By symmetry, any of the n boxes can be empty)

[Alternatively, Ans $= P(\text{Box\# is empty} \mid \text{only one Box is empty})$

$$= \frac{P(\text{only Box \#1 is empty})}{P(\text{only one box is empty})} = \frac{(b)}{(c)}$$

$$= \frac{1}{n}.]$$

(Ross. p. 358 #70, 71, 72)

3. An urn contains a large number of coins. Each coin gives a head with probability p. The value of p varies from coin to coin but is uniformly distributed in $[0, 1]$. Now a coin is selected at random. This *same* coin is used in the following question.

   (a) The coin is tossed once. What is the probability that the outcome is a head?

   (b) The coin is tossed twice. What is the probability that both outcomes are heads?

   (c) The coin is tossed n times. Let X be the number of heads obtained. Find the distribution of X, i.e. find $P(X = i)$.

   (d) The coin is kept being tossed until a head is obtained. Let N be the number of tossing needed. Find the distribution of N, i.e. find $P(N = n)$.

   (e) Find $E(N)$, the expectation of N.

   Note: The following integration identity might be useful: for any positive integers a, b,

$$\int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \frac{(a-1)!(b-1)!}{(a+b-1)!}.$$

(a) $P(H) = \int_0^1 P(H/p)\,P(p)\,dp = \int_0^1 p\,dp = \frac{1}{2}$

$P(p) = 1$

(b) $P(HH) = \int_0^1 P(HH/p)\,P(p)\,dp = \int_0^1 p^2\,dp = \frac{1}{3}$

(c) $P(X=i) = \int_0^1 \binom{n}{i} p^i (1-p)^{n-i}\,dp$

$$= \binom{n}{i} \frac{i!\,(n-i)!}{(n+1)!} = \frac{1}{(n+1)}$$

(d) $P(N=n) = \int_0^1 q^{n-1} p\,dp$

$$= \int_0^1 p(1-p)^{n-1}\,dp$$

You can use this blank page.

$$= \frac{1!\,(n-1)!}{(n+1)!} = \frac{1}{(n+1)n}$$

(e) $E(N) = \sum_{n=1}^{\infty} n P(N=n)$

$$= \sum_{n=1}^{\infty} n \cdot \frac{1}{(n+1)n}$$

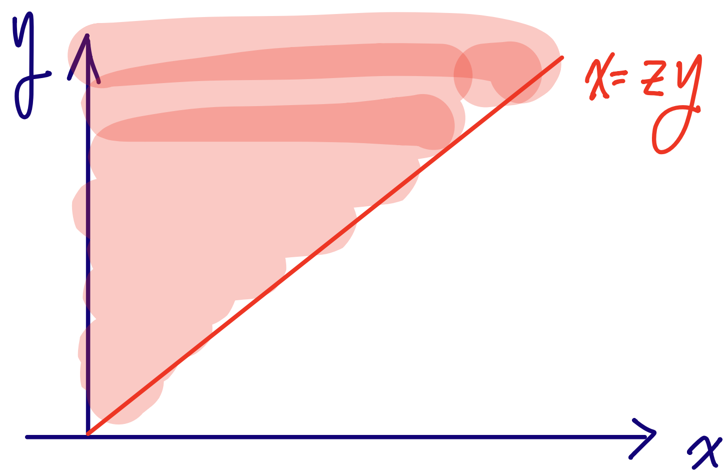
$$= \sum_{n=1}^{\infty} \frac{1}{n+1}$$

$$= \infty$$ (Harmonic series, divergent)

4. (a) Let X and Y be two identical, independent exponentially distributed random variables with parameter 1. Find the pdf of $Z = \frac{X}{Y}$.

(b) Let X and Y be two identical, independent standard normal random variables with parameter 1. Find the pdf of $Z = \frac{X}{Y}$.

(a) $P(Z \le z) = P\left(\frac{X}{Y} \le z\right) = P(X \le zY)$



$$= \int_0^\infty \int_0^{\frac{x}{z}} e^{-x} e^{-y}\, dy\, dx = \int_0^\infty e^{-x}\left(e^{-\frac{x}{z}}\right) dx$$

cdf of Z

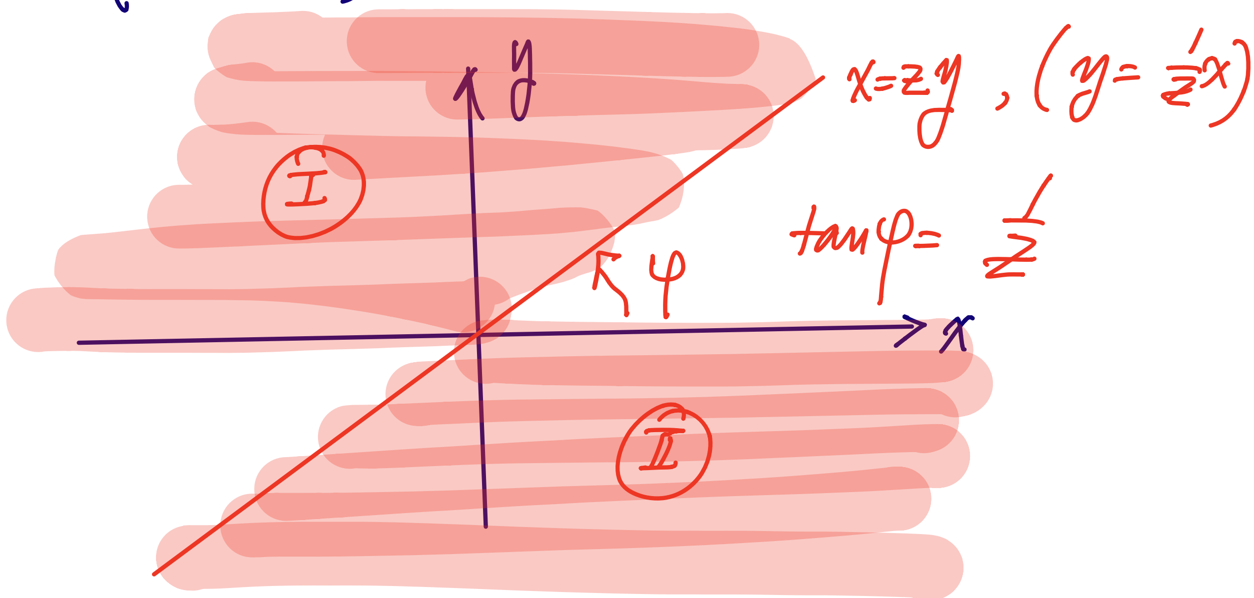
$$= \int_0^\infty e^{-(1+\frac{1}{z})x}\, dx = \frac{1}{1+\frac{1}{z}} = \frac{z}{1+z}$$

$$p_Z(z) = \frac{d}{dz}\left(\frac{z}{1+z}\right) = \frac{1+z-z}{(1+z)^2} = \frac{1}{(1+z)^2}, \quad z \geq 0$$

(a) $P(Z \le z) = P\left(\frac{X}{Y} \le z\right)$

You can use this blank page.

$$= P(X \le zY,\ Y \ge 0) + P(X \ge zY,\ Y < 0)$$



$$= \iint_{\mathrm{I}} \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}\,dx\,dy + \iint_{\mathrm{II}} \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}\,dx\,dy$$

$$= 2\iint_{\mathrm{I}} \frac{e^{-\frac{x^2+y^2}{2}}}{2\pi}\,dx\,dy$$ (Use Polar coord: $dx\,dy = r\,dr\,d\theta$)

$$= 2\int_{\varphi}^{\pi} \underbrace{\int_0^{\infty} \frac{e^{-r^2/2}}{2\pi}\,r\,dr}_{\frac{1}{2\pi}}\,d\theta = \frac{1}{\pi}\int_{\varphi}^{\pi} d\theta = \frac{\pi-\varphi}{\pi}$$

You can use this blank page.

$$= \frac{\pi - \tan^{-1}\frac{1}{z}}{\pi}$$

cdf of Z

$$p_Z(z) = \frac{d}{dz}\left(\frac{\pi - \tan^{-1}\frac{1}{z}}{\pi}\right)$$

$$= \frac{1}{\pi}\left(-\frac{1}{1+\frac{1}{z^2}}\right)\left(-\frac{1}{z^2}\right)$$

$$= \frac{1}{\pi(z^2+1)}$$

$\sim$ Cauchy Distribution.

5. (a) Let X and Y be bi-variate normal random variables with joint probability density given as follows:

$$p(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right\}$$

Find the conditional probability density of X and Y, i.e. find,

$$p_{X|Y}(x|y).$$

Relate your answer to some common distribution – be as quantitative as possible.

(b) Let $X \sim \mathcal{N}(\Theta, 1)$, i.e. normal random variable with mean Θ and variance 1. Now the actual value of Θ is not known but is distributed as $\mathcal{N}(0,1)$, i.e. normal random variable with mean 0 and variance 1. The above information is expressed as:

$$p_{X|\Theta}(x|\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\theta)^2}{2}\right), \quad \text{and} \quad p_\Theta(\theta) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\theta^2}{2}\right).$$

An experiment is performed and an actual value X is obtained. Find the conditional probability distribution of Θ given $X = x$, i.e. find

$$p_{\Theta|X}(\theta|x).$$

Relate your answer to some common distribution – be as quantitative as possible.

(a) $$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}}}{2\pi\sqrt{1-\rho^2}\, p(y)}$$

$$= C(y)\, e^{-\frac{x^2 - 2\rho xy}{2(1-\rho^2)}}$$

$$= C(y)\, e^{-\frac{x^2 - 2\rho xy + \rho^2 y^2 - \rho^2 y^2}{2(1-\rho^2)}}$$

$$= C(y)\, e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}} \sim N(\rho y, 1-\rho^2)$$

(MUST Be $\frac{1}{\sqrt{2\pi(1-\rho^2)}}$)

(b) $p(\theta|x) = \dfrac{p(\theta,x)}{p(x)} = \dfrac{p(x|\theta)\,p(\theta)}{p(x)}$

You can use this blank page.

$$= C(x)\, e^{-\frac{(x-\theta)^2}{2}}\, e^{-\frac{\theta^2}{2}}$$

$$= C(x)\, e^{-\frac{x^2-2x\theta+2\theta^2}{2}}$$

$$= C(x)\, e^{-(\theta^2 - x\theta)}$$

$$= C(x)\, e^{-(\theta^2 - x\theta + \frac{x^2}{4} - \frac{x^2}{4})}$$

$$= C(x)\, e^{-(\theta-\frac{x}{2})^2}$$

$$= C(x)\, e^{-\frac{(\theta-\frac{x}{2})^2}{2(\frac{1}{2})}} \quad \sim N\left(\frac{x}{2}, \frac{1}{2}\right)$$

MUST BE $\dfrac{1}{\sqrt{2\pi\left(\frac{1}{2}\right)}} = \dfrac{1}{\sqrt{\pi}}$