MA/STAT 519: Introduction to Probability
Spring 2018, Final Examination

Instructor: Yip

- This test booklet has SEVEN QUESTIONS, totaling 100 points for the whole test. Questions #5, 6, 7 are worth 20 points each. You have 120 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**. One two-sided 8×11 formula sheet is allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.
- When hints are provided in the question, it is at your discretion if you want to utilize them.

Name: Answer Key (Department:)

Question	Score
1.(10 pts)	
2.(10 pts)	
3.(10 pts)	
4.(10 pts)	
5.(20 pts)	
6.(20 pts)	
7.(20 pts)	
Total (100 pts)	

1. Consider infinitely many iid experiments of throwing a pair of dice. The outcome of each experiment is the sum of the two numbers. Let $N \geq 1$ be the number of experiments such that the number 5 or 7 *first* appears as the outcome. Let further X be that number at the N-th experiment, i.e. X is either a 5 or 7.

   (a) Find $P(N = n)$ for $n \geq 1$. (Ross, p. 50 #25, p104 #76)
   (Hint: Find first $P(N \geq n)$.)

   (b) Find $P(X = 5)$, equivalently, find the probability that 5 appears before 7.
   (Hint: Condition on the first outcome.)

$$P(5) = P\big((1,4),(2,3),(3,2),(4,1)\big) = \frac{4}{36}$$

$$P(7) = P\big((1,6),(2,5),\cdots,(6,1)\big) = \frac{6}{36}$$

$$P(5 \text{ or } 7) = \frac{10}{36}, \quad P(\text{not } 5, \text{ not } 7) = \frac{26}{36}$$

(a) $P(N=n) = P(1^{st}\ n-1 \text{ outcomes} \neq 5, 7,\ n^{th} \text{ outcome} = 5 \text{ or } 7)$

(Geometric r.v.) $= \left(\frac{26}{36}\right)^{n-1}\left(\frac{10}{36}\right) = \left(\frac{13}{18}\right)^{n-1}\left(\frac{5}{18}\right)$

(b) Intuitively, the answer should be $\frac{P(5)}{P(5)+P(7)}$.

$$P(X=5) = P(X=5 \mid 1^{st}=5)\,P(1^{st}=5)$$
$$+ P(X=5 \mid 1^{st}=7)\,P(1^{st}=7)$$
$$+ P(X=5 \mid 1^{st}\neq 5,7)\,P(1^{st}\neq 5,7)$$

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$$P(X=5) = 1 \times P(5) + 0 \times P(7) + P(X=5)\,P(\text{not } 5, \text{not } 7)$$

combined.

$$P(X=5)\,[\underbrace{1 - P(\text{not } 5, \text{not } 7)}_{P(5 \text{ or } 7)}] = P(5)$$

Hence $$P(X=5) = \frac{P(5)}{P(5 \text{ or } 7)}$$

$$= \frac{P(5)}{P(5)+P(7)}$$

$$= \frac{4}{4+6} = \frac{2}{5}$$

2. Let X and Y be two independent discrete random variables. For each of the following cases, compute the conditional distribution of X given $X+Y$, i.e. find

$$P\Big(X=i\Big|X+Y=j\Big)$$

Also, relate the conditional distribution to some common, i.e. well known, distribution.

Ross p249 Example (4b)

(a) X is Poisson with parameter λ and Y is Poisson with parameter μ.

(b) X is Binomial with parameter n and p and Y is Binomial with parameter m and p.

Ross p 275 #14

(c) X and Y are Geometric with parameter p.

X & Y are ind.

$$P(X=i \mid X+Y=j)=\frac{P(X=i,X+Y=j)}{P(X+Y=j)}=\frac{P(X=i)P(Y=j-i)}{P(X+Y=j)}$$

a) $X\sim \text{Poisson}(\lambda)$, $Y=\text{Poisson}(\mu)$, $X+Y\sim \text{Poisson}(\lambda+\mu)$

$$P(X=i \mid X+Y=j)=\frac{\dfrac{e^{-\lambda}\lambda^i}{i!}\,\dfrac{e^{-\mu}\mu^{j-i}}{(j-i)!}}{\dfrac{e^{-(\lambda+\mu)}(\lambda+\mu)^j}{j!}}$$

$$=\frac{j!}{i!(j-i)!}\,\frac{\lambda^i\mu^{j-i}}{(\lambda+\mu)^j} \quad \sim \text{Bin}\left(j,\frac{\lambda}{\lambda+\mu}\right)$$

$$=\binom{j}{i}\left(\frac{\lambda}{\lambda+\mu}\right)^i\left(\frac{\mu}{\lambda+\mu}\right)^{j-i}$$

(b) $X \sim Bin(n,p)$, $Y \sim Bin(m,p)$, $X+Y \sim Bin(n+m,p)$

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$$P(X=i \mid X+Y=j) = \frac{\binom{n}{i} p^i q^{n-i} \binom{m}{j-i} p^{j-i} q^{m-j+i}}{\binom{m+n}{j} p^j q^{m+n-j}}$$

$$= \frac{\binom{n}{i}\binom{m}{j-i}}{\binom{m+n}{j}} \overset{\mathcal{D}}{\sim} \text{Hypergeometric}$$

(c) $X \sim Geom(p)$, $Y \sim Geom(p)$, $X+Y \sim NegBin(2,p)$

$$P(X=i \mid X+Y=j) = \frac{p q^{i-1} p q^{j-i-1}}{\binom{j-1}{2-1} p^2 q^{j-2}} = \frac{1}{j-1}$$

$$\sim unif(1,2,\dots,j-1)$$

Note: ① The answer does not depend on i!

② $\sum_{i=1}^{j-1} P(X=i \mid X+Y=j) = 1$

3. Ten balls are to be distributed among 5 urns, with each ball independently going into urn i with probability p_i. Let X_i be the number of balls that go into urn i. Note that we have $\sum_{i=1}^{5} p_i = 1$ and $\sum_{i=1}^{5} X_i = 10$.

(a) Find the joint distribution of $X_i, X_2, \ldots X_5$, i.e. find

$$P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4 = x_4, X_5 = x_5).$$

(b) Find the (marginal) distribution of X_1.

(c) Find the distribution of $X_1 + X_2$.

(d) Find the distribution $X_1 + X_2 + X_3$.

(e) Find $P(X_1 + X_2 = 4 | X_1 + X_2 + X_3 = 7)$.

(Ross, p174 #24)

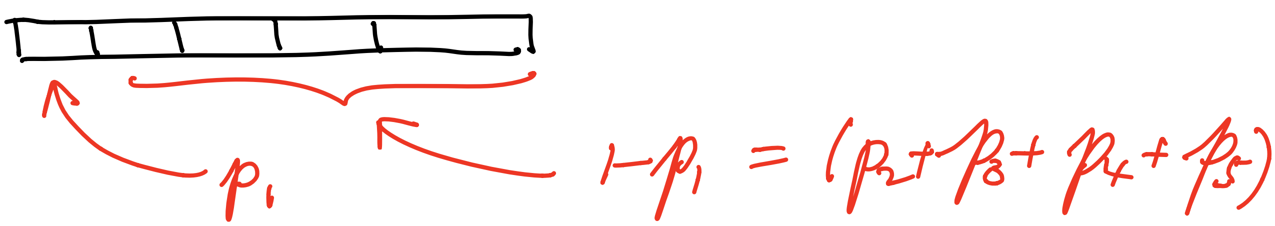
(Hint: You can do Parts (b)–(e) without knowing Part (a).)

Multinomial dist.

(a) $P(X_1 = x_1, \cdots, X_5 = x_5)$

$$= \frac{10!}{x_1! x_2! x_3! x_4! x_5!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4} p_5^{x_5}$$

(b)



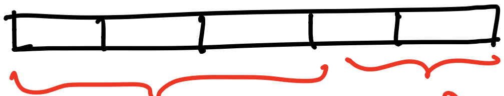
$$P(X_1 = i) = \binom{10}{i} p_1^i (1-p_1)^{10-i} \sim Bin(10, p_1)$$

(c)

$$P(X_1+X_2=i)=\binom{10}{i}(p_1+p_2)^i(1-p_1-p_2)^{10-i}$$

You can use this scrap paper.

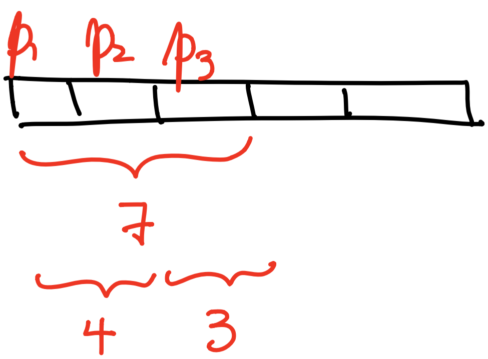
(d)



$p_1+p_2+p_3$ $\quad$ $1-(p_1+p_2+p_3)=p_4+p_5$

$$P(X_1+X_2+X_3=i)=\binom{10}{i}(p_1+p_2+p_3)^i(p_4+p_5)^{10-i}$$

(e) $P(X_1+X_2=4 \mid X_1+X_2+X_3=7)$



$$=\binom{7}{4}\left(\frac{p_1+p_2}{p_1+p_2+p_3}\right)^4\left(\frac{p_3}{p_1+p_2+p_3}\right)^3$$

[Alternatively,

$$P[X_1+X_2=4 \mid X_1+X_2+X_3=7]=\frac{P(X_1+X_2=4,\ X_1+X_2+X_3=7)}{P(X_1+X_2+X_3=7)}$$

$$=\frac{P(X_1+X_2=4,\ X_3=3,\ X_4+X_5=3)}{P(X_1+X_2+X_3=7)}$$

$$= \frac{\frac{10!}{4!\,3!\,3!}\,(p_1+p_2)^4\,p_3^{\,3}\,(p_4+p_5)^3}{\frac{10!}{7!\,3!}\,(p_1+p_2+p_3)^7\,(p_4+p_5)^3}$$

$$= \frac{7!}{4!\,3!}\left(\frac{p_1+p_2}{p_1+p_2+p_3}\right)^4\left(\frac{p_3}{p_1+p_2+p_3}\right)^3. \;]$$

4. Let X_1, X_2, X_3 be three iid continuous random variables. Find the following probabilities.

   (a) $P(X_1 < X_2)$

   (b) $P(X_1 < X_2 < X_3)$;

   (c) $P(X_1 > X_2 | X_1 > X_3)$;

   (d) $P(X_1 > X_2 | X_1 < X_3)$;

   (e) $P(X_1 > X_2 | X_2 < X_3)$.

   (Hint: Use symmetry argument. The continuity of the random variables are only used to *exclude* the events $\{X_1 = X_2\}$, $\{X_2 = X_3\}$, or $\{X_1 = X_3\}$ which have probability zero. Hence there is no distinction between the events $\{X_1 < X_2\}$ and $\{X_1 \leq X_2\}$ and so forth.)

(Ross p. 276 #19)

Note: By symmetry, $P(X_i < X_j) = P(X_j < X_i)$ $(i \neq j)$

Hence $P(X_i < X_j) = \frac{1}{2}$

Similarly, all the events $\{X_i < X_j < X_k\}$ $(i \neq j \neq k)$ are equally likely. Since there are 6 $(= 3!)$ of them, hence,

$P(X_i < X_j < X_k) = \frac{1}{6}$

(a) $P(X_1 < X_2) = \frac{1}{2}$

You can use this scrap paper.

(b) $P(X_1 < X_2 < X_3) = \frac{1}{6}$

(c) $P(X_1 > X_2 \mid X_1 > X_3) = \dfrac{P(X_1 > X_2, X_1 > X_3)}{P(X_1 > X_3)}$

$= \dfrac{P(\max = X_1)}{P(X_1 > X_3)} = \dfrac{1/3}{1/2} = 2/3$

(Any of X_1, X_2, X_3 is equally likely to be the max.)

(d) $P(X_1 > X_2 \mid X_1 < X_3)$

$= \dfrac{P(X_1 > X_2, X_1 > X_3)}{P(X_1 < X_3)} = \dfrac{P(X_3 < X_1 < X_2)}{P(X_1 < X_3)}$

$= \dfrac{1/6}{1/2} = 1/3$

(e) $P(X_1 > X_2 / X_2 < X_3)$

$$= \frac{P(X_1 > X_2,\ X_2 < X_3)}{P(X_2 < X_3)}$$

$$= \frac{P(\min = X_2)}{P(X_2 < X_3)}$$

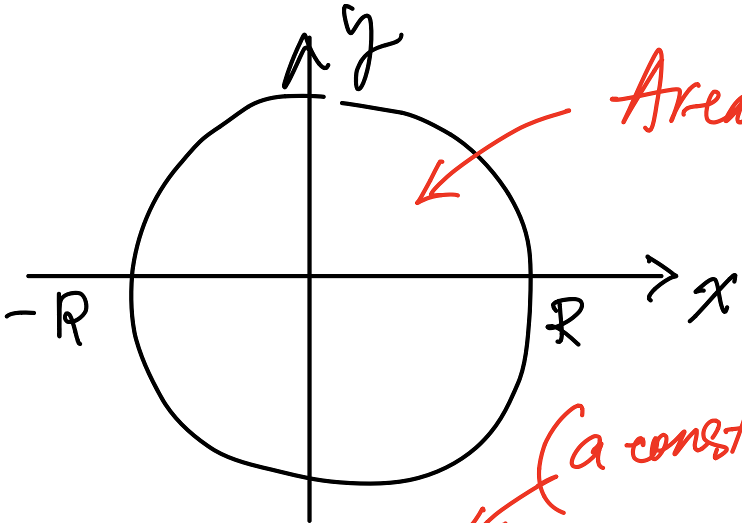
(Any of X_1, X_2, X_3 is equally likely to be the min.)

$$= \frac{1/3}{1/2} = 2/3$$

5. Consider a circle of radius R, and suppose that a point (X, Y) inside the circle is chosen randomly with uniform distribution.

Ross, p.225 Example (1d)

   (a) Find the joint pdf of X and Y.
   (b) Find the (marginal) pdfs of X and Y.
   (c) Find the pdf of $D = \sqrt{X^2 + Y^2}$, i.e. D is the distance of the point (X, Y) to the origin.
   (d) Find $E(D)$, the expectation of D.

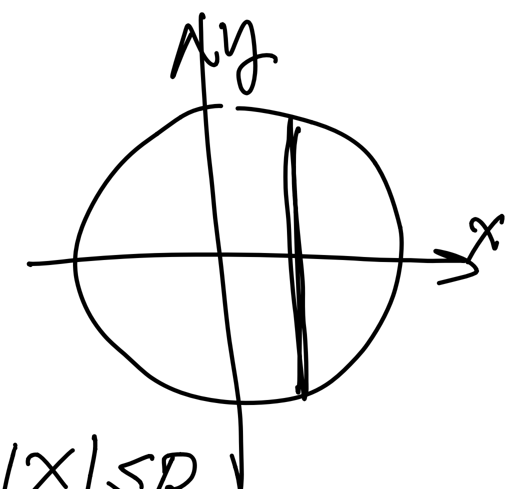


Area $= \pi R^2$

(a constant)

(a) $$p(x,y) = \begin{cases} \dfrac{1}{\pi R^2} & x^2+y^2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) $$p_X(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} p(x,y)\,dy$$

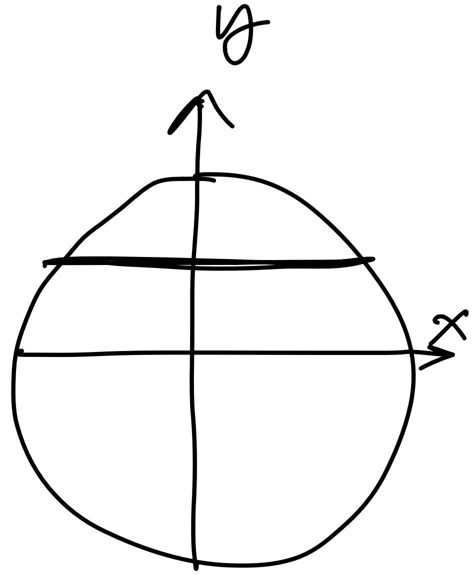


$$= \frac{2\sqrt{R^2-x^2}}{\pi R^2}, \quad |x| \le R$$

You can use this scrap paper.

Similarly

$$p_Y(y) = \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} p(x,y)\,dx$$



$$= \frac{2\sqrt{R^2-y^2}}{\pi R^2}, \quad |y| \leq R$$

(c) $P(D \leq r) = P(x^2+y^2 \leq r^2) = \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2}$

$$p_D(r) = \frac{d}{dr} P(D \leq r) = \frac{2r}{R^2}$$

(d) $E(D) = \int_0^R r \frac{2r}{R^2}\,dr = \frac{1}{R^2} \frac{2}{3} r^3 \Big|_0^R$

$$= \frac{2}{3} R$$

6. A Gamma random variable with parameters α and λ has density given by:

$$p(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, \quad \text{for } x \geq 0.$$

The mean and variance of the above random variable are given by $\frac{\alpha}{\lambda}$ and $\frac{\alpha}{\lambda^2}$. For the remaining parts, let X_i's be a sequence of iid standard normal random variables.

(a) Find the pdf of X_1^2 and relate it to Gamma distribution. *Derive your answer, even though you know/remember the answer.*

(b) Find the exact pdf of $X_1^2 + X_2^2 + \cdots X_n^2$.

(c) Give an approximation of $P(80 \leq X_1^2 + \cdots X_{100}^2 \leq 120)$. Express your answer in terms of the cdf of a standard normal.

(a) $Y = X_1^2$

$$P(Y \leq y) = P(X_1^2 \leq y) = P(-\sqrt{y} < X_1 < \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dy = 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$P_Y(y) = \frac{d}{dy} P(Y \leq y) = 2\frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{d}{dy}\sqrt{y}$$

$$= \frac{2}{\sqrt{2\pi}} \frac{1}{2} y^{-\frac{1}{2}} e^{-\frac{y}{2}} = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y}$$

$\sim$ Gamma$(\frac{1}{2}, \frac{1}{2}) = \chi^2$ with 1-deg of freedom!

$$\left[p_Y(y) = \frac{1}{\sqrt{2\pi}} \left(2 \tfrac{1}{2} y\right)^{-\frac{1}{2}} e^{-\frac{1}{2}y} \right.$$

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$$= \frac{1}{\sqrt{2\pi}} \, 2 \times \frac{1}{2} \times 2^{-\frac{1}{2}} \left(\frac{1}{2} y\right)^{-\frac{1}{2}} e^{-\frac{1}{2}y}$$

$$= \frac{1}{\sqrt{\pi}} \, \frac{1}{2} \left(\frac{1}{2} y\right)^{\frac{1}{2}-1} e^{-\frac{1}{2}y}, \quad \alpha = \frac{1}{2}, \ \lambda = \frac{1}{2}$$

$$\left. \left(\text{also } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}\right) \right]$$

(b) $X_1^2 + X_2^2 + \cdots + X_n^2$

$\sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right) + \cdots + \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$

$\sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$, ($\chi^2$ with n-deg of freedom.)

(c) Use Central Limit Theorem.

$S_n = X_1^2 + X_2^2 + \cdots + X_n^2$ $\quad \alpha = \frac{1}{2}, \ \lambda = \frac{1}{2}$

$E(S_n) = n E X_1^2 = n\left(\frac{\alpha}{\lambda}\right) = n$

$$\mathrm{Var}(S_n) = n\,\mathrm{Var}(X_1) = n\frac{\alpha}{\lambda^2} = 2n$$

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$$\left(\text{Alternatively} = \mathrm{Var}\left(\mathrm{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)\right) = \frac{n/2}{(1/2)^2} = 2n.\right)$$

$$P(80 \leq S_n \leq 120)$$

$$= P\left(\frac{80 - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}} \leq \frac{S_n - ES_n}{\sqrt{\mathrm{Var}(S_n)}} \leq \frac{120 - E(S_n)}{\sqrt{\mathrm{Var}(S_n)}}\right)$$

$$= P\left(\frac{80 - n}{\sqrt{2n}} \leq Z \leq \frac{120 - n}{\sqrt{2n}}\right) \quad (n = 100)$$

$$= P\left(-\frac{20}{\sqrt{200}} \leq Z \leq \frac{20}{\sqrt{200}}\right)$$

$$= P(-\sqrt{2} \leq Z \leq \sqrt{2}) = \Phi(\sqrt{2}) - \Phi(-\sqrt{2})$$

$$= 2\Phi(\sqrt{2}) - 1.$$

(Note. There is an "exact" answer for (d).

Since $X_1^2 + X_2^2 + \cdots + X_n^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$,

$$P\left(80 \leq X_1^2 + \cdots + X_n^2 \leq 120 \right)$$

$$= \int_{80}^{120} \frac{\frac{1}{2}\left(\frac{1}{2}x\right)^{\frac{n}{2}-1} e^{-\frac{1}{2}x}}{\Gamma\left(\frac{n}{2}\right)}\, dx .$$

But the above is hard to integrate!)

7. Consider the following signal(X)+noise(Y)=observation(Z) model:

$$X \sim \mathcal{N}(\mu_1, \sigma_1^2), \quad Y \sim \mathcal{N}(0, \sigma_2^2), \quad Z = X + Y.$$

Assume that X and Y are independent. The goal is to say something about X upon observing Z.

(a) Find the joint pdf of X and Z. (Hint: Consider the map (X, Y) to $(X, X+Y)$.

(b) Find the (marginal) pdf of $Z = X + Y$.

(c) Find the conditional pdf of X given Z, i.e. $p(x|z)$. Relate this conditional pdf to some common distribution.

(d) Find the conditional expectation and variance of X, given $Z = z$, i.e. find

$$E(X|Z = z) = \int x p(x|z)\, dx, \text{ and } Var(X|Z = z) = E(X^2|Z = z) - (E(X|Z = z))^2.$$

(Hint: For Part (c), concentrate on the "symbolic form" of the distribution. The explicit constant(s) will come out "naturally". Part (d) should be very transparent once you have the correct answer to Part (c).)

(a) $\begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow \begin{pmatrix} X \\ Z = X+Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$

$$p(x, z) = g(x, y) \frac{1}{\left|\frac{dx\,dz}{dx\,dy}\right|} = g(x,y) \cdot \left(\frac{1}{1}\right)$$

$$= \frac{e^{-\frac{(x-\mu)^2}{2\sigma_1^2}} e^{-\frac{y^2}{2\sigma_2^2}}}{2\pi \sigma_1 \sigma_2}$$

$(y = z - x)$

$$= \frac{e^{-\frac{(x-\mu)^2}{2\sigma_1^2}} e^{-\frac{(z-x)^2}{2\sigma_2^2}}}{2\pi \sigma_1 \sigma_2}$$

(b) $Z = X + Y = N(\mu, \sigma_1^2) + N(0, \sigma_2^2)$

You can use this scrap paper.

$$= N(\mu, \sigma_1^2 + \sigma_2^2)$$

$$p_Z(z) = \frac{e^{-(z-\mu)^2 / 2(\sigma_1^2+\sigma_2^2)}}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}$$

(c) $p(x/z) = p(x,z) / p_Z(z)$

$$= \frac{\frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{(x-\mu)^2}{2\sigma_1^2}} e^{-\frac{(z-x)^2}{2\sigma_2^2}}}{\frac{e^{-(z-\mu)^2/2(\sigma_1^2+\sigma_2^2)}}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}}$$

(Note: it does not matter what $p_Z(z)$ is)

You can use this scrap paper.

$$= C(z)\, e^{-\frac{(x-\mu)^2}{2\sigma_1^2} - \frac{(z-x)^2}{2\sigma_2^2}}$$

some constant, which does not involve x!

$$= C(z)\exp\left\{-\frac{x^2-2\mu x+\mu^2}{2\sigma_1^2} - \frac{z^2-2zx+x^2}{2\sigma_2^2}\right\}$$

$$= C(z)\exp\left\{-\frac{1}{2}\left(\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)x^2 - 2\left(\frac{\mu}{\sigma_1^2}+\frac{z}{\sigma_2^2}\right)x + \left(\frac{\mu^2}{\sigma_1^2}+\frac{z^2}{\sigma_2^2}\right)\right)\right\}$$

(it does not matter.)

$$= C(z)\cdot \exp\left\{-\frac{1}{2}\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right)\left(x^2 - 2\,\frac{\mu\sigma_2^2+z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\,x\right)\right\}$$

Complete the square!

$$= C(z) \times \exp\left\{-\frac{1}{2}\left(\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\right)\left(x-\frac{\mu\sigma_2^2+z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2\right\}$$

$$= C(z)\, e^{-\frac{\left(x-\frac{\mu\sigma_2^2+z\sigma_1^2}{\sigma_1^2+\sigma_2^2}\, z\right)^2}{2\left(\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)}}$$

$$\sim N\left(\frac{\mu\sigma_2^2+z\sigma_1^2}{\sigma_1^2+\sigma_2^2},\ \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)$$

($C(z)$ must be $\dfrac{1}{\sqrt{2\pi\,\dfrac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}}}$.)

(d) $E[X|Z_i] = \dfrac{\mu\sigma_2^2 + Z\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$

$Var(X|Z_i) = \dfrac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2}.$