MA/STAT 519: Introduction to Probability
Fall 2018, Mid-Term Examination

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- This test booklet has FOUR QUESTIONS, totaling 80 points for the whole test. You have 75 minutes to do this test. **Plan your time well. Read the questions carefully.**
- This test is **closed book, with no electronic device**. One two-sided-8×11 formula sheet is allowed.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible way** how you arrive at them.

Name: Answer Key (Department:)

Question	Score
1.(20 pts)	
2.(20 pts)	
3.(20 pts)	
4.(20 pts)	
Total (80 pts)	

1. Let X and Y be two independent discrete random variables. For each of the following cases, compute the conditional distribution of X given $X+Y$, i.e. find

$$P\Big(X=i\Big|X+Y=j\Big)$$

If possible, relate the conditional distribution to some common, i.e. well known, distribution.

(a) X is Poisson with parameter λ and Y is Poisson with parameter μ.

(b) X is Binomial with parameter n and p and Y is Binomial with parameter m and p.

(c) X and Y are Geometric with parameter p.

$$P(X=i/X+Y=j) = \frac{P(X=i, X+Y=j)}{P(X+Y=j)}$$

$$= \frac{P(X=i, Y=j-i)}{P(X+Y=j)}$$

$$= \frac{P(X=i)P(Y=j-i)}{P(X+Y=j)} \quad (*)$$

(a) $X \sim Poisson(\lambda)$, $Y \sim Poisson(\mu)$

$X+Y \sim Poisson(\lambda+\mu)$

$$(*) = \frac{\cancel{e^{-\lambda}}\lambda^i}{i!} \frac{\cancel{e^{-\mu}}\mu^{j-i}}{(j-i)!} \Big/ \frac{\cancel{e^{-(\lambda+\mu)}}(\lambda+\mu)^j}{j!}$$

$$= \frac{j!}{i!(j-i)!} \frac{\lambda^i \mu^{j-i}}{(\lambda+\mu)^j} = \binom{j}{i}\left(\frac{\lambda}{\lambda+\mu}\right)^i \left(\frac{\mu}{\lambda+\mu}\right)^{j-i}$$

$$\sim Bin\left(j, \frac{\lambda}{\lambda+\mu}\right)$$

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(b)

$$\underbrace{\overbrace{n \text{ trials}}^{X \sim Bin(n,p)} \;\; \overbrace{m\text{-trials}}^{Y \sim Bin(m,p)}}_{X+Y \sim Bin(n+m,\, p)}$$

$$X+Y \sim Bin(n+m, p).$$

$$(*) = \frac{\binom{n}{i}\binom{m}{j-i}}{\binom{m+n}{j}} \qquad (\text{Hypergeometric})$$

(c) $X \sim Geom(p)$, $\quad Y \sim Geom(p)$

$$X+Y \sim NegBin(2, p)$$

$$(*) = \frac{\cancel{q^{i-1}}\,\cancel{p}\;\cancel{q^{j-i-1}}\,\cancel{p}}{\binom{j-1}{2-1}\cancel{q^{j-2}}\,\cancel{p^2}} = \frac{1}{j-1}$$

$\downarrow$

(uniform, does not depend on i)

2. Suppose n balls are distributed at random into r boxes in such a way that each ball chooses a box independently of each other. Let S be the number of *empty boxes*. Compute ES and $Var(S)$.

(Hint: Consider the random variables X_i (for $i = 1, 2, \ldots, r$) which equals 1 if the i-th box is empty and 0 otherwise. Related S and the X_i's.)

$$S = X_1 + X_2 + \cdots + X_r$$

$$ES = EX_1 + EX_2 + \cdots + EX_r$$

$$EX_1 = 1 \times P(X_1 = 1) + 0 \times P(X_1 = 0)$$

$$= \underbrace{P(X_1 = 1)}_{\text{Box\#1 is empty}} = \left(\frac{r-1}{r}\right)^n$$

no. of balls

no. of boxes to choose from

total no. of boxes

Hence $ES = r\left(\frac{r-1}{r}\right)^n$

$$E(S^2) = E\left(\sum_{i=1}^{r} X_i\right)^2$$

$$= E\left(\sum_i X_i^2 + \sum_{i \neq j} X_i X_j\right)$$

$$= \sum_i E(X_i^2) + \sum_{i \neq j} E(X_i X_j)$$

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- Since $X_i = 0, 1$, $X_i^2 = X_i$

$$E(X_i^2) = EX_i = \left(\frac{r-1}{r}\right)^n$$

no. of boxes to choose from

- $$EX_iX_j = P(\underbrace{X_i = 1, X_j = 1}_{\text{Box \#i, \#j are empty}}) = \left(\frac{r-2}{r}\right)^n$$

no. of balls

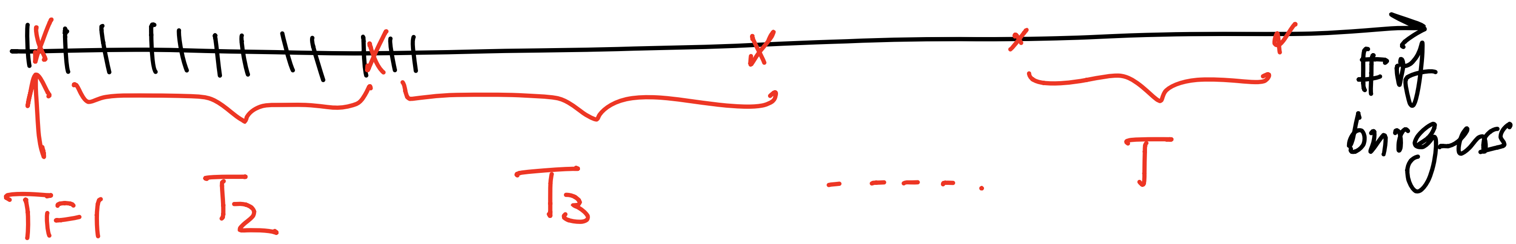
total no. of boxes

Hence $$E(S^2) = r\left(\frac{r-1}{r}\right)^n + r(r-1)\left(\frac{r-2}{r}\right)^n$$

$$Var(S) = E(S^2) - (ES)^2$$
$$= r\left(\frac{r-1}{r}\right)^n + r(r-1)\left(\frac{r-2}{r}\right)^n - \left[r\left(\frac{r-1}{r}\right)^n\right]^2$$

3. McDonald's newest promotion is putting a toy inside every one of its hamburger. Suppose there are N distinct types of toys and each of them is equally likely to be put inside any of the hamburger. What is the expected value and variance of the number of hamburgers you need to order (or eat) before you have a complete set of the N toys.

(Hint: consider the number of hamburgers you need to order (or eat) in between getting one and two dinstinct types of toys, two and three distinct types of toys, and so forth.)



$$\left.\begin{aligned} T_1 &= 1 \\ T_2 &= \text{geom}\left(\frac{N-1}{N}\right) \\ T_3 &= \text{geom}\left(\frac{N-2}{N}\right) \\ &\vdots \\ T_N &= \text{geom}\left(\frac{1}{N}\right) \end{aligned}\right\} \text{ independent geom. r.v.'s}$$

Let S = Total no. of burgers purchased.

Then $S = T_1 + T_2 + \cdots + T_N$

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$$E(\text{geom}(p)) = \frac{1}{p}, \quad Var(\text{Geom}(p)) = \frac{q}{p^2}$$

$$ES = ET_1 + ET_2 + \cdots + ET_N$$

$$= 1 + \frac{N}{N-1} + \frac{N}{N-2} + \cdots + \frac{N}{1}$$

$$= N\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right)$$

$$Var(S) = Var(T_1) + Var(T_2) + \cdots + Var(T_N)$$

(Since the T_i's are independent)

$$= 0 + \frac{\frac{1}{N}}{\left(\frac{N-1}{N}\right)^2} + \frac{\frac{2}{N}}{\left(\frac{N-2}{N}\right)^2} + \cdots + \frac{\frac{N-1}{N}}{\left(\frac{1}{N}\right)^2}$$

$$= \frac{N}{(N-1)^2} + \frac{2N}{(N-2)^2} + \cdots + \frac{(N-1)N}{1^2}$$

$$= N\left[\frac{1}{(N-1)^2} + \frac{2}{(N-2)^2} + \cdots + \frac{N-1}{(1)^2}\right]$$

4. Consider a box with M white balls and N black balls. You are asked to get n balls (at random) out of the box without replacement. Let X be the number of white balls obtained.

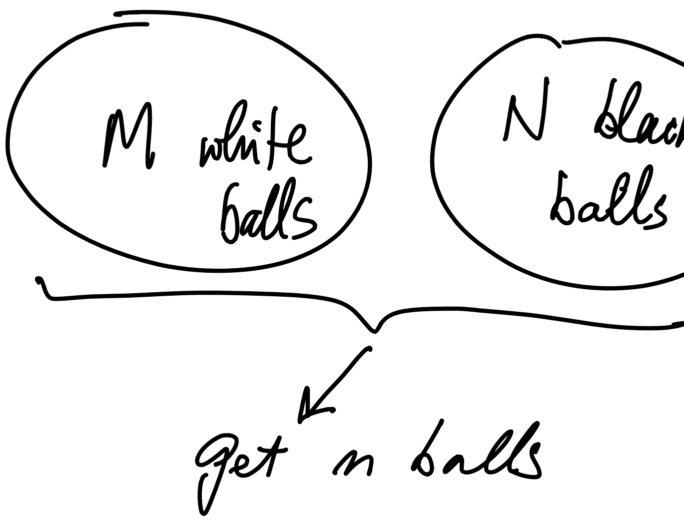
   (a) Find the probability distribution of X.

   (b) Find the expectation and variance of X.

   (Hint: imagine that you get the balls sequentially, one by one. Introduce the indicator functions $X_i = 1$ for $i = 1, \ldots n$ defined as X_i equals one if i-th ball is white and zero otherwise. Relate X and the X_i's.)

   (c) Now suppose M and N tends to infinity such that $\frac{M}{M+N} \longrightarrow p$. Derive and identify the limiting probability distribution of X.

   (d) Under the same limiting procedure, find the limiting expectation and variance of X.



(a) $$P(X=i) = \frac{\binom{M}{i}\binom{N}{n-i}}{\binom{M+N}{n}}$$ (Hypergeometric)

(b) $$X = X_1 + X_2 + \cdots + X_n$$

$$E(X) = EX_1 + EX_2 + \cdots + EX_n$$

$$EX_1 = P(X_1 = 1) = \frac{M}{M+N}$$

no. of white balls

total no. of balls

You can use this blank page.

Hence

$$EX = nEX_1 = \frac{nM}{M+N}$$

the i^{th} & j^{th} balls are white.

$$E(X^2) = \sum_i EX_i^2 + \sum_{i \neq j} E X_i X_j$$

$$= \sum_i P(X_i = 1) + \sum_{i \neq j} \underbrace{P(X_i = 1, X_j = 1)}$$

$$= \sum_i \frac{M}{M+N} + \sum_{i \neq j} \frac{M(M-1)}{(M+N)(M+N-1)}$$

$$= \frac{nM}{M+N} + n(n-1)\frac{M(M-1)}{(M+N)(M+N-1)}$$

$$Var(X) = E(X^2) - (EX)^2$$

$$= \frac{nM}{M+N} + \frac{n(n-1)M(M-1)}{(M+N)(M+N-1)} - \left(\frac{nM}{M+N}\right)^2$$

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(c) $$P(X=i) = \frac{\binom{M}{i}\binom{N}{n-i}}{\binom{M+N}{n}}$$

$$= \frac{\frac{M(M-1)\cdots(M-i+1)}{i!}\;\frac{N(N-1)(N-2)\cdots(N-n+i+1)}{(n-i)!}}{\frac{(M+N)(M+N-1)\cdots\cdots(M+N-n+1)}{n!}}$$

$$= \frac{n!}{i!(n-i)!}\,\frac{\times\left(M(M-1)\cdots(M-i+1)\right) \times \left(N(N-1)\cdots(N-n+i+1)\right)}{(M+N)(M+N-1)\cdots\cdots(M+N-n+1)}$$

i terms

n-i terms

n terms

$$= \binom{n}{i}\left(\frac{M}{M+N}\right)\left(\frac{M-1}{M+N-1}\right)\left(\frac{M-2}{M+N-2}\right)\cdots\left(\frac{M-i+1}{M+N-i+1}\right)\times$$

$$\times\left(\frac{N}{M+N-i}\right)\left(\frac{N-1}{M+N-i-1}\right)\cdots\cdots\left(\frac{N-n+i+1}{M+N-n+1}\right)$$

$$\xrightarrow[\frac{M}{N+N}\to p]{M,N\to+\infty} \binom{n}{i} p^i q^{n-i}$$

$$\frac{M}{M+N}\to p$$

$$\frac{N}{M+N}\to 1-p=q$$

$$\sim Bin(n,p)$$

Note: The answer also makes intuitive sense as $M, N \to +\infty$, it does not make any difference whether it is with or without replacement.

(d) Hence

$$EX \to np, \quad Var(X) \to npq$$

The above can also be checked directly from the formula:

$$EX = \frac{nM}{M+N} \to np$$

$$Var(X) = \frac{nM}{M+N} + \frac{n(n-1)M(M-1)}{(M+N)(M+N-1)} - \left(\frac{nM}{M+N}\right)^2$$

$$\to np + n(n-1)p^2 - n^2p^2$$

$$= np - np^2 = np(1-p) = npq.$$