STAT 519 Fall 2019
Introduction to Probability

Midterm Exam

- You can use a calculator, although it may not be very helpful.
- You have 120 minutes.
- Show your work.
- In order to get full credits, you need to give **correct** and **simplified** answers and explain in a **comprehensible** way how you arrive at them.
- GOOD LUCK!

Name:

Problem 1[15 points] Apeach has \$120 and wants to donate some or all of them to 4 different charities. The donation should be in units of \$10. How many ways can he donate his money?

Solution: Dividing all quantities by 10, we are looking for the number of positive integers

(1) x_1, x_2, x_3, x_4 such that $\sum_{i=1}^{4} x_i = 12$.

The short answer is

$$\binom{12+4-1}{4-1} = \binom{15}{3} = \underline{455}$$

Justification:

(i) The # of solutions to (1) is the same as the # of solutions to

(2) $y_1, \dots, y_4$ s.t. $y_i \geq 1$ and $\sum_{i=1}^{4} y_i = 16$

(ii) The problem (2) is equivalent to the classical stars & bars problem

16 stars - 3 bars to be placed in the corresponding 15 intervals

$\hookrightarrow \binom{15}{3}$ solutions

See pb 33 - ch 1 - 9th ed

Problem 2[15 points] Find the probability of "Triple" (Exactly three same numbers/letters) in the 5 card poker.

Solution: We take

$S = \{5 \text{ cards hands}\}$. Thus $|S| = \binom{52}{5}$.

We consider the uniform probability on S

Then

$\#\{\text{hands of the form } a\,a\,a\,b\,c\}$

$$= \underset{\substack{\uparrow\\ \text{pick}\\ a \text{ among}\\ 13}}{\binom{13}{1}} \times \underset{\substack{\uparrow\\ \text{colors}\\ \text{for } a}}{\binom{4}{3}} \times \underset{\substack{\uparrow\\ \text{pick}\\ bc \text{ among } 12}}{\binom{12}{2}} \times \underbrace{4 \times 4}_{\text{colors for } b\,c}$$

$$P(\text{triple}) = \frac{13 \times 4 \times \binom{12}{2} \times 16}{\binom{52}{5}} \simeq 2.1\%$$

Problem 3[20 points] The BIG 10 league has 14 schools in spite of its name. 6 teams are considered as "contenders" and other 8 teams are considered as "pretenders". If a team is a contender, the probability that the team goes to a bowl game is 0.6. If a team is a pretender, the probability that the team goes to a bowl game is 0.3. What is the (conditional) probability that a team goes to a bowl game in the second year when the team also went to a bowl game in the first year?

Solution:

Set C = Contender
Pr = Pretender
B_i = Goes to bowl on year i

Data: $P(C) = \frac{3}{7}$ $P(Pr) = \frac{4}{7}$ $P(B_i|C) = .6$ $P(B_i|Pr) = .3$

We wish to compute $P(B_2|B_1)$.

Solution 1

$$P(B_2|B_1) = \underbrace{P(B_2|C\,B_1)}_{=P(B_2|C)}P(C|B_1) + P(B_2|Pr\,B_1)\,P(Pr|B_1)$$

$$= P(B_2|C)\,\frac{P(B_1|C)P(C)}{P(B_1)} + P(B_2|Pr)\,\frac{P(B_1|Pr)\,P(Pr)}{P(B_1)} \quad (1)$$

Then $P(B_1) = P(B_1|C)P(C) + P(B_1|Pr)P(Pr)$

$$= 0.6 \times \frac{3}{7} + 0.3 \times \frac{4}{7} = \frac{3}{7}$$

Thus

$$P(B_2|B_1) = \frac{7}{3}\left\{(0.6)^2 \times \frac{3}{7} + (0.3)^2 \times \frac{4}{7}\right\} =$$

$$= 48\%$$

Solution 2

$$P(B_2|B_1) = \frac{P(B_2B_1)}{P(B_1)} = \frac{P(B_2B_1|C)P(C)}{P(B_1)} + \frac{P(B_2B_1|Pr)P(Pr)}{P(B_1)}$$

$$\stackrel{(\mathrm{cdt} \perp\!\!\!\perp)}{=} \frac{P(B_2|C)\,P(B_1|C)P(C)}{P(B_1)} + \frac{P(B_2|Pr)\,P(B_1|Pr)\,P(Pr)}{P(B_1)}$$

↳ same formula as (1)

Problem 4[20 points] You can use only three axioms of the probability in this problem.
(a) State the countable additivity.
(b) Show that the probability of an empty set is 0.
(c) Using (a),(b), show the finite additivity.

Solution:

(a) If $\{A_i; i \geq 1\}$ are mutually exclusive

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(b) Axiom 2 is $P(S) = 1$. Then

$$1 = P(S \cup \emptyset) = P(S) + P(\emptyset) = 1 + P(\emptyset)$$

We get $P(\emptyset) = 0$

(c) Let $\{A_i; 1 \leq i \leq n\}$ mutually exclusive and $\{B_i; i \geq 1\}$ with $B_i = A_i$ for $i \leq n$, $B_i = \emptyset$ for $i > n$

Then B_i's are mutually exclusive and

(i) $P\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left(\bigcup_{i=1}^{n} A_i\right)$

(ii) $P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i) = \sum_{i=1}^{n} P(A_i) + \sum_{i=n+1}^{\infty} P(\emptyset)$

$$= \sum_{i=1}^{n} P(A_i)$$

We get $P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$

Problem 5[20 points] Suppose that there are 5 types of coupons and that each new coupon collected is, independent of previous selections, a type i coupon with probability $p_i = ki$ for some constant k, $\sum_{i=1}^{5} p_i = 1$. Supposed that 8 coupons are to be collected. If A_i is the event that there is at least one type i coupon among those collected, find
(a) $P(A_2 \cup A_3)$
(b) $P(A_2|A_3)$

Solution:

(i) Compute k: $k = \left(\sum_{i=1}^{5} i\right)^{-1} = \frac{1}{15}$

(ii) $P(A_2) = 1 - P(A_2^c) = 1 - (1-p_2)^8 = 1 - \left(\frac{13}{15}\right)^8 = 0.682$

$P(A_3) = 1 - P(A_3^c) = 1 - (1-p_3)^8 = 1 - \left(\frac{12}{15}\right)^8 = 0.832$

(a) $P(A_2 \cup A_3) = 1 - P((A_2 \cup A_3)^c) = 1 - P(A_2^c A_3^c)$

$= 1 - (1 - p_2 - p_3)^8 = 1 - \left(\frac{10}{15}\right)^8 = 1 - \left(\frac{2}{3}\right)^8 = 0.961$

(iii) $P(A_2 A_3) = P(A_2) + P(A_3) - P(A_2 \cup A_3)$

$= 1 - \left(\frac{13}{15}\right)^8 - \left(\frac{12}{15}\right)^8 + \left(\frac{2}{3}\right)^8 = 0.553$

(b) $P(A_2 | A_3) = \dfrac{P(A_2 A_3)}{P(A_3)} = \dfrac{0.553}{0.832} = 0.665$

See Example 4i - Chapter 3 - 9th edition

Problem 6[20 points] The number of customers coming to Frodo and Neo's ice cream shop satisfies Poisson assumptions and the rate is 6 customers per hour. They open their shop at 11 am. If there are more than 2 customers in first 10 minutes after opening, they provide a free scoop to the third customer. October 1 is Tuesday, and they open everyday ~~except Sundays.~~

(a) Find the probability that they provide a free scoop in any given business day.
(b) What is the probability that they provide a free scoop for the third time on Oct 10?
(c) What is the probability that they provide a free scoop 10 times in the month of October?

Solution:

(a) $X =$ # customers on a given day for 1st 10 minutes

Then $X \sim \mathcal{P}(\lambda)$ $\lambda = 1$

$P(\text{free scoop}) = P(X > 2) = 1 - \sum_{i=0}^{2} P(X=i) = 1 - e^{-1}(1 + 1 + \frac{1}{2}) = 1 - \frac{5}{2}e^{-1}$
$= p \equiv 0.08$

(b) Bernoulli trial with proba of success p
$Y =$ instant of 3rd success. Then $Y \sim NBin(3, p)$
we wish to find

$P(Y = 10) = \binom{9}{2} p^3 (1-p)^7 = 0.01 = 1\%$

(c) $Z =$ # free scoops in October

Then $Z \sim Bin(n, p)$

$n = 31$ $p = 0.08$

$P(Z = 10) = \binom{31}{10} 0.08^{10} (0.92)^{21}$

$= 8.5 \times 10^{-5}$

Problem 7[20 points] Show that X is a Poisson random variable with parameter λ, then

$$E[X^n] = \lambda E(X+1)^{n-1}. \quad , \text{ for } n \geq 1$$

Find $E[X^3]$ using the above result.

Solution:

(i)
$$E[X^n] = e^{-\lambda} \sum_{k=0}^{\infty} k^n \frac{\lambda^k}{k!}$$
$$= e^{-\lambda} \sum_{k=1}^{\infty} k^{n-1} k \frac{\lambda^k}{k!}$$
$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} k^{n-1} \frac{\lambda^{k-1}}{(k-1)!}$$
$$= \lambda e^{-\lambda} \sum_{j=0}^{\infty} (j+1)^{n-1} \frac{\lambda^j}{j!}$$
$$= \lambda E[(X+1)^{n-1}]$$

(ii)

$n=1$: $E[X^1] = \lambda$

$n=2$: $E[X^2] = \lambda E[X+1] = \lambda^2 + \lambda = \lambda(\lambda+1)$

$n=3$:
$$E[X^3] = \lambda E[(X+1)^2]$$
$$= \lambda E[X^2 + 2X + 1]$$
$$= \lambda [\lambda^2 + \lambda + 2\lambda + 1]$$
$$= \lambda(\lambda^2 + 3\lambda + 1)$$

Problem 8[20 points] X follows Negative Binomial(r, p). Find $E[X]$ and $Var[X]$.

Solution:

See solution on slides
or Ross' book