MA 532 - Midterm - Spring 24

Problem 1. We consider a random walk X on $\mathbb{Z}^2$. Namely $X_0 = (0,0)$, and for $n \geq 1$ we have

$$X_n = \sum_{k=1}^{n} Z_k \,,$$

where $\{Z_k;\, k \geq 1\}$ is a sequence of independent and identically distributed random variables with

$$\mathbf{P}\left(Z_k = (-1,0)\right) = \mathbf{P}\left(Z_k = (1,0)\right) = \mathbf{P}\left(Z_k = (0,-1)\right) = \mathbf{P}\left(Z_k = (0,1)\right) = \frac{1}{4} \,.$$

Since both X_n and Z_k take values in $\mathbb{Z}^2$, we will write

$$X_n = \left(X_n^1, X_n^2\right), \quad \text{and} \quad Z_k = \left(Z_k^1, Z_k^2\right).$$

1.1. For a given $k \geq 1$, prove that Z_k^1 and Z_k^2 are not independent.

Definition of independence Z_k^1, Z_k^2 both take values in $\{-1,1\}$. Hence

$$Z_k^1 \perp\!\!\!\perp Z_k^2$$

$$\Longleftrightarrow$$

$$\mathbb{P}\left(\left(Z_k^1, Z_k^2\right) = (i,j)\right) = \mathbb{P}\left(Z_k^1 = i\right)\, \mathbb{P}\left(Z_k^2 = j\right),$$

for all $i,j \in \{0,1\}$

Marginals of z_k^1, z_k^2 we have

$$\mathbb{P}(z_k^1 = 1) = \mathbb{P}(z_k = (1,0)) = \frac{1}{4}$$

$$\mathbb{P}(z_k^1 = -1) = \mathbb{P}(z_k = (-1,0)) = \frac{1}{4}$$

$$\mathbb{P}(z_k^1 = 0) = \mathbb{P}(z_k = (0,-1)) + \mathbb{P}(z_k = (0,1)) = \frac{1}{2}$$

we also find

$$\mathbb{P}(z_k^2 = -1) = \mathbb{P}(z_k^2 = 1) = \frac{1}{4}$$

$$\mathbb{P}(z_k^2 = 0) = \frac{1}{2}$$

Independence we have for instance

$$\mathbb{P}(z_k^1 = 0, z_k^2 = 0) = 0$$

$$\mathbb{P}(z_k^1 = 0)\, \mathbb{P}(z_k^2 = 0) = \frac{1}{4}$$

Thus

$$\mathbb{P}(z_k^1 = 0, z_k^2 = 0) \neq \mathbb{P}(z_k^1 = 0)\, \mathbb{P}(z_k^2 = 0)$$

and $\boxed{z_k^1 \not\perp\!\!\!\perp z_k^2}$

1.2. For a given $k \geq 1$, we set

$$Y_k^1 = Z_k^1 + Z_k^2, \quad \text{and} \quad Y_k^2 = Z_k^1 - Z_k^2.$$

Prove that (Y_k^1, Y_k^2) is a couple of independent random variables such that for $j = 1, 2$ we have

$$\mathbf{P}\left(Y_k^j = -1\right) = \mathbf{P}\left(Y_k^j = 1\right) = \frac{1}{2}.$$

Table The following table summarizes the distribution of Z_k, together with the values of $Y_k^1 = Z_k^1 + Z_k^2$ and $Y_k^2 = Z_k^1 - Z_k^2$

$Z_k^1 \backslash Z_k^2$	-1	0	1
-1	0	$1/4$ -1 -1	0
0	$1/4$ -1 1	0	$1/4$ 1 -1
1	0	$1/4$ 1 1	0

Table for Y_k From the previous table we deduce the table for Y_k:

$Y_k^1 \setminus Y_k^2$	-1	1	marg Y_k^1
-1	$1/4$	$1/4$	$\frac{1}{2}$
1	$1/4$	$1/4$	$\frac{1}{2}$
marg Y_k^2	$\frac{1}{2}$	$\frac{1}{2}$	

From the table we see that

$$\mathbb{P}(Y_k = (i,j)) = \mathbb{P}(Y_k^1 = i)\, \mathbb{P}(Y_k^2 = j),$$

for all $i,j \in \{-1,1\}$. Hence

$$\boxed{Y_k^1 \perp\!\!\!\perp Y_k^2}$$

1.3. We now set
$$U_n = X_n^1 + X_n^2, \quad \text{and} \quad V_n = X_n^1 - X_n^2.$$
Show that $U = \{U_n;\, n \geq 1\}$ and $V = \{V_n;\, n \geq 1\}$ are two independent symmetric random walks starting at 0.

Expression in terms of Y_k we have

$$U_n = X_n^1 + X_n^2$$
$$= \sum_{k=1}^{n} Z_k^1 + \sum_{k=1}^{n} Z_k^2$$
$$= \sum_{k=1}^{n} \left(Z_k^1 + Z_k^2\right)$$
$$= \sum_{k=1}^{n} Y_k^1$$

U is a symmetric rw From the previous table, Y_k^1 is such that

$$\mathbb{P}(Y_k^1 = -1) = \mathbb{P}(Y_k^1 = 1) = \frac{1}{2}$$

The Y_k's are $\perp\!\!\!\perp$. Hence U is a symmetric random walk.

Result for V Similarly to U, we have

$$V_n = \sum_{k=1}^{n} Y_k^2$$

Hence V is a symmetric r.w.

Furthermore, since $Y^1 \perp\!\!\!\perp Y^2$ we get

$$\boxed{U \perp\!\!\!\perp V}$$

1.4. Let T_1 be the random time defined by

$$T_1 = \inf\{n \geq 1;\, U_n = 1\}.$$

Prove that the probability generating function G_1 of T_1 has the expression

$$G_1(s) = \frac{1 - (1 - s^2)^{1/2}}{s}.$$

One cannot use directly the result from class, the above formula has to be carefully proven. However, one can resort to the following identity: if $T_2 = \inf\{n \geq 1;\, U_n = 2\}$, and we set

$$f_1(n) = \mathbf{P}(T_1 = n), \quad f_2(n) = \mathbf{P}(T_2 = n), \quad F_1(s) = \sum_{n=1}^{\infty} f_1(n) s^n, \quad F_2(s) = \sum_{n=1}^{\infty} f_2(n) s^n,$$

then we have

$$F_2(s) = (F_2(s))^2.$$

Solution:

<u>Conditioning on X_1</u> For $n > 1$ we have

$$f_1(n) = \mathbb{P}(T_1 = n)$$

$$= \overbrace{\tfrac{1}{2}\mathbb{P}(T_1 = n \mid X_1 = 1)}^{=0 \text{ if } n>1} + \tfrac{1}{2}\mathbb{P}(T_1 = n \mid X_1 = -1)$$

$$= \tfrac{1}{2} f_2(n-1)$$

Moreover, $f_1(1) = \mathbb{P}(X_1 = 1) = \tfrac{1}{2}$

Summing over n for $s \in [0,1)$ we have

$$F_1(s) = \sum_{n=0}^{\infty} f_1(n) s^n = \frac{1}{2} s + \sum_{n=2}^{\infty} f_1(n) s^n$$

$$= \frac{1}{2} s + \sum_{n=2}^{\infty} \frac{1}{2} f_2(n-1) s^n$$

$$= \frac{1}{2} s + \frac{1}{2} s \sum_{m=1}^{\infty} f_2(m) s^m$$

$$= \frac{1}{2} s \left(1 + F_2(s) \right)$$

Equation for F_1 we have seen

$$F_1(s) = \frac{1}{2} s \left(1 + F_2(s) \right).$$

Since we are given the relation $F_2(s) = F_1(s)^2$, we get

$$s(F_1(s))^2 - 2F_1(s) + s = 0$$

Solving for $F_1(s)$ $F_1(s)$ is solution of the quadratic equation

$$s x^2 - 2x + s = 0$$

we get roots with the quadratic formula:

$$\Delta = 4 - 4s^2 = 4(1-s^2)$$

$$x = \frac{2 \mp 2(1-s^2)^{\frac{1}{2}}}{2s}$$

we pick the root which converges as $s \longrightarrow 0$. we obtain

$$\boxed{F_1(s) = \frac{1-(1-s^2)^{\frac{1}{2}}}{s}}$$

1.5. Let D_1 be the line

$$D_1 = \inf\left\{(x,y) \in \mathbb{R}^2;\, x+y=1\right\}.$$

We wish to get some information about the random time

$$\hat{T}_1 = \inf\{n \geq 1;\, X_n \in D_1\}.$$

Prove that $\hat{T}_1 = T_1$, and deduce an expression for the probability generating function $\hat{G}_1$ of the random variable $\hat{T}_1$.

Expression with U_n we have

$$\hat{T}_1 = \inf\{n \geq 1;\ X_n \in D_1\}$$

$$= \inf\{n \geq 1;\ \overbrace{X_n^1 + X_n^2}^{= U_n} = 1\}$$

$$= \inf\{n \geq 1;\ U_n = 1\}$$

$$= T_1$$

Identity for $\hat{G}_1$ Since $T_1 = \hat{T}_1$, in particular we have

$$T_1 \stackrel{(d)}{=} \hat{T}_1,$$

i.e T_1 & $\hat{T}_1$ have the same pmf. Therefore

$$\boxed{\hat{G}_1 = F_1}$$

1.6. Show that $\hat{T}_1$ is a finite random variable, namely

$$\mathbf{P}\left(\hat{T}_1 = \infty\right) = 0.$$

We have

$$P(\hat{T}_1 < \infty) = \hat{G}_1(1)$$

$$= F_1(1)$$

$$= \left. \frac{1 - (1-s^2)^{\frac{1}{2}}}{s} \right|_{s=1}$$

$$= 1$$

Thus

$$\boxed{\mathbb{P}(\hat{T}_1 = \infty) = 0}$$

Problem 2. Let $\{U_n; n \geq 1\}$ be a sequence representing successive dice rolls. That is the U_n's are independent uniform random variables in $\{1, \ldots, 6\}$. Prove that the following processes are Markov chains and specify their transition matrix.

2.1. $X_n \equiv$ largest roll U_j shown up to n-th roll.

Dynamics for X we have

$$X_n = \max\{U_j ; 1 \leq j \leq n\}$$

Thus

$$X_{n+1} = \max\{X_n , U_{n+1}\}$$

$$X_{n+1} = \varphi(X_n, U_{n+1}),$$

where $\{U_n; n \geq 1\}$ i.i.d in $\{1, .., 6\}$
and

$$\varphi(x, u) = \max\{x, u\}$$

Thus

$$\boxed{X \text{ Markov chain}}$$

State space : $S = \{1, .., 6\}$

Transition matrix we have

$$p_{ij} = \mathbb{P}(\varphi(i, U_1) = j)$$

$$= \mathbb{P}(\max(i, U_1) = j)$$

Hence for $i \in \{1, .., 6\}$ we can only have $j \in \{i, .., 6\}$ such that $p_{ij} > 0$. Next since

$$U_1 \sim \mathcal{U}(\{1, ..., 6\})$$

we get

$$p_{ii} = \mathbb{P}(\max(i, U_1) = i)$$

$$p_{ii} = \mathbb{P}(U_1 \leq i)$$

$$\boxed{p_{ii} = \frac{i}{6}}$$ $(=1 \text{ if } i = 6)$

and for $j \in \{i+1, .., 6\}$, and $i \leq 5$

$$p_{ij} = \mathbb{P}(\max(i, U_1) = j) = \mathbb{P}(U_1 = j)$$

$$\boxed{p_{ij} = \frac{1}{6}}$$

2.2. $Y_n \equiv$ Number of sixes in the first n rolls.

Dynamics for Y we have

$$Y_{n+1} = Y_n + \mathbb{1}_{(U_n=6)}$$

$$Y_{n+1} = \varphi(Y_n, U_{n+1}),$$

where

$$\varphi(x,u) = x + \mathbb{1}_{(u=6)}$$

Thus

$$\boxed{X \text{ Markov chain}} \quad \text{on } S = \mathbb{N}$$

Transition matrix we have

$$p_{ij} = \mathbb{P}(\varphi(i, U_1) = j)$$

$$= \mathbb{P}(i + \mathbb{1}_{(U_1=6)} = j)$$

$$= \mathbb{P}(\mathbb{1}_{(U_1=6)} = j-i)$$

$$\boxed{p_{ij} = \frac{1}{6}\, \mathbb{1}_{(j=i+1)} + \frac{5}{6}\, \mathbb{1}_{(j=i)}}$$