## MA 532- Midtern - Spring 24

**Problem 1.** We consider a random walk X on  $\mathbb{Z}^2$ . Namely  $X_0 = (0,0)$ , and for  $n \ge 1$  we have

$$X_n = \sum_{k=1}^n Z_k$$

where  $\{Z_k; k \ge 1\}$  is a sequence of independent and identically distributed random variables with

$$\mathbf{P}(Z_k = (-1,0)) = \mathbf{P}(Z_k = (1,0)) = \mathbf{P}(Z_k = (0,-1)) = \mathbf{P}(Z_k = (0,1)) = \frac{1}{4}.$$

Since both  $X_n$  and  $Z_k$  take values in  $\mathbb{Z}^2$ , we will write

$$X_n = (X_n^1, X_n^2)$$
, and  $Z_k = (Z_k^1, Z_k^2)$ .

**1.1.** For a given  $k \ge 1$ , prove that  $Z_k^1$  and  $Z_k^2$  are not independent.

Definition of independence Zi, Zi both take values in <-1,15. Hence

2, 11 22

 $P(2_{k}, t_{k}^{2}) = (i,j) = P(t_{k}^{2} = i) P(t_{k}^{2} = j),$ 

for all i, E < 0, 15

Marginals of z', z' we have  $P(t_{k} = 1) = P(t_{k} = (1,0)) = \frac{1}{4}$  $P(z_{k} = -1) = P(z_{k} = (-1, 0)) = \frac{1}{4}$  $\mathbb{P}(z_{k}=0) = \mathbb{P}(z_{k}=(0,-1)) + \mathbb{I}(z_{k}=(0,1)) = \frac{1}{2}$ we also find  $P(t_k^2 = -1) = P(t_k^2 = 1) = \frac{1}{4}$  $P(t_{k}^{2}=0)=\frac{1}{2}$ Independence we have for instance  $P(2_{k}^{\prime}=0,2_{k}^{2}=0)=0$  $P(t_{k} = 0) P(t_{k} = 0) = \frac{1}{4}$ Thus  $P(t_{k}=0, t_{k}=0) \neq P(t_{k}=0) P(t_{k}=0)$ Zi # Zi and

**1.2.** For a given  $k \ge 1$ , we set

$$Y_k^1 = Z_k^1 + Z_k^2$$
, and  $Y_k^2 = Z_k^1 - Z_k^2$ .

Prove that  $(Y_k^1, Y_k^2)$  is a couple of independent random variables such that for j = 1, 2 we have

$$\mathbf{P}(Y_k^j = -1) = \mathbf{P}(Y_k^j = 1) = \frac{1}{2}$$

Table The following table summarizes the distribution of  $z_k$ , together with the values of  $Y'_k = z'_k + z'_k$  and  $Y'_k = z'_k - z''_k$ 

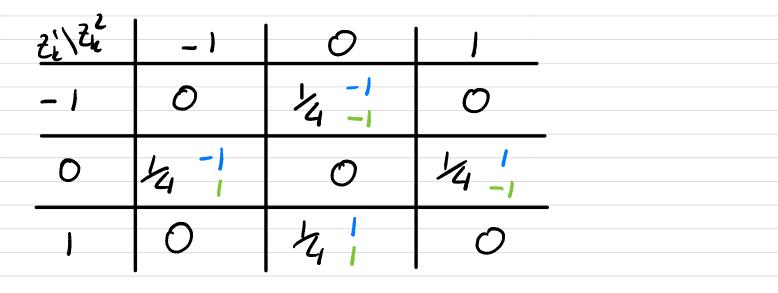


Table for Yn From the previous rable we deduce the table for Yk:

Y, Yh Mang Y' -14 4 2 14 と 12 Tany Y. 19

the table we see that From

 $P(Y_{k} = (i,j)) = P(Y_{k} = i) P(Y_{k} = j),$ for all  $i, j \in 2-1, 15$ . Hence

Y' 11 Y'2

1.3. We now set

$$U_n = X_n^1 + X_n^2$$
, and  $V_n = X_n^1 - X_n^2$ .

Show that  $U = \{U_n; n \ge 1\}$  and  $V = \{V_n; n \ge 1\}$  are two independent symmetric random walks starting at 0.

× ne)in in terms of Y, we have

 $U_n = X_n' + X_n^2$  $\sum_{k=1}^{n} \frac{z_{k}^{\prime}}{z_{k}^{\prime}} + \sum_{k=1}^{n} \frac{z_{k}^{2}}{z_{k}^{\prime}}$  $\sum_{k=1}^{n} \left( \frac{z_{k}}{k} + \frac{z_{k}^{2}}{k} \right)$ Y'

) is a symmetric ru From the merious table, X' is such that

 $P(Y_{k} = -1) = P(Y_{k} = 1) = \frac{1}{2}$ 

The Y's are IL. Hence U symmetric random walk a

Realt for V Similarly to U, we

have

 $V_n = \sum_{k=1}^n \frac{1}{k^2}$ 

Visa symmetric Rw. Hence Furthermae, since Y'11 Ye we get



**1.4.** Let  $T_1$  be the random time defined by

$$T_1 = \inf \{ n \ge 1; U_n = 1 \}$$

Prove that the probability generating function  $G_1$  of  $T_1$  has the expression

$$G_1(s) = \frac{1 - (1 - s^2)^{1/2}}{s}$$

One cannot use directly the result from class, the above formula has to be carefully proven. However, one can resort to the following identity: if  $T_2 = \inf\{n \ge 1; U_n = 2\}$ , and we set

$$f_1(n) = \mathbf{P}(T_1 = n), \quad f_2(n) = \mathbf{P}(T_2 = n), \quad F_1(s) = \sum_{n=1}^{\infty} f_1(n)s^n, \quad F_2(s) = \sum_{n=1}^{\infty} f_2(n)s^n,$$

then we have

$$F_2(s) = (F_2(s))^2$$
.

Solution:

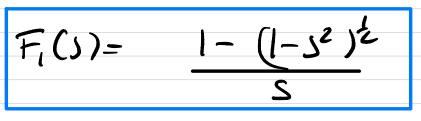
conditioning on X, Fil n>1 we have  $f_i(n) = IP(T_i = n) = 0 \quad i \neq n > 1$  $=\frac{1}{2}P(T_{1}=n|X_{1}=1)+\frac{1}{2}P(T_{1}=n|X_{1}=-1)$  $= \frac{1}{2} f_2(n-1)$ Mueover,  $f_{1}(1) = \mathbb{P}(X_{1}=1) = \frac{1}{2}$ 

<u>Summing over n</u> Fn SETO, I) we have  $F_{1}(s) = \sum_{n=0}^{\infty} f_{1}(n) s^{n} = \frac{1}{2} s + \sum_{n=0}^{\infty} f_{1}(n) s^{n}$  $=\frac{1}{2}J + \sum_{n=2}^{\infty} \frac{1}{2} \frac{1}{4} (n-1) S^{n}$  $=\frac{1}{2}S + \frac{1}{2}S \sum_{m=1}^{\infty} \frac{1}{4}(m)S^{m}$  $= \frac{1}{2}S(1+F_{2}(S))$ 

Equation for F, we have sen  $F_{1}(s) = \frac{1}{2}S(1 + F_{2}(s)).$ Since we are given the relation  $F_2(s) = F_1(s)^2$ , we get

 $S(F_{r}(s))^{2} - 2F_{r}(s) + S = 0$ 

Solving for F.(s) F.(s) is solution of the quadratic equation  $Sx^{2} - 2x + 3 = 0$ we get nots with the quadratic frmula:  $\Delta = 4 - 4 s^{2} = 4(1 - s^{2})$  $\mathcal{X} = \frac{2}{7} \frac{2}{7} \frac{2}{(1-3^2)^2}$ 2 S we pick the root which converges as s->0. Ne obtain



**1.5.** Let  $D_1$  be the line

1.

$$D_1 = \inf \{ (x, y) \in \mathbb{R}^2; x + y = 1 \}.$$

We wish to get some information about the random time

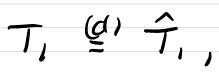
 $\hat{T}_1 = \inf \{ n \ge 1; X_n \in D_1 \}.$ 

Prove that  $\hat{T}_1 = T_1$ , and deduce an expression for the probability generating function  $\hat{G}_1$  of the random variable  $\hat{T}_1$ .

Expension with Un we have

 $\hat{\tau}_{i=}$  inf(n>i;  $x_n \in \mathcal{P}_i$ )  $inf \langle n \geq i; \quad X_n + X_n^2 = 1 \rangle$ inf (nzi; Un =1)

Identity fu  $\hat{G}_{i}$ , Since  $T_{i}=\hat{T}_{i}$ , in particular we have





**1.6.** Show that  $\hat{T}_1$  is a finite random variable, namely

$$\mathbf{P}\left(\hat{T}_1 = \infty\right) = 0.$$

we have

 $\mathcal{P}(\hat{\tau},<\infty) = \hat{G}_{i}(1)$ 

F(1)

 $= (-)^{2}$ 



 $\tilde{T}_{1} = \infty$ ) = 0

**Problem 2.** Let  $\{U_n; n \ge 1\}$  be a sequence representing successive dice rolls. That is the  $U_n$ 's are independent uniform random variables in  $\{1, \ldots, 6\}$ . Prove that the following processes are Markov chains and specify their transition matrix.

**2.1.**  $X_n \equiv \text{largest roll } U_i \text{ shown up to } n\text{-th roll.}$ 

Dynamics fux we have X = max LU; ; ISJENS Thus max {Xn, Unry X AH =  $X_{n+1} = \mathcal{Q}(X_n, U_{n+1}),$ where {Un; n21} i.i.d in {1,..., 65 and  $\varphi(x, u) = max \{ x, u \}$ Thus × Markov chain

State space:  $S = \{1, \dots, 6\}$ 

Transition matrix we have  $P_{ij} = P(\varphi(i, U_i) = j)$  $= \mathbb{P}(\max(i, U, ) = j)$ Hence for  $i \in \{1, ..., 6\}$  we can only have  $j \in \{1, ..., 6\}$  such that Pij > 0. Next since U,~ U(31,..,65) we get =  $\mathbb{P}(\max(i, v, i) = i)$ Pii  $\rho_{ii} = P(U_{i} \leq i)$  $P_{ii} = \frac{i}{6} \quad (=1 \quad if \quad i=6)$ and for jes it,...,65, and i 55  $P_{ij} = \mathbb{P}(\text{mode}(i, U_i) = j) = \mathbb{P}(U_i = j)$  $li_{\sigma} = \frac{1}{6}$ 

**2.2.**  $Y_n \equiv$  Number of sixes in the first *n* rolls.

Dynamics for X we have

Ynn= Yn + 1(Un=6)

YnH = U(Yn, Unn),

where

 $\Psi(x,u) = x + 1_{(u=6)}$ 

Thus

X Markov chain On S= IN

Transition matrix we have

 $P_{ij} = \mathbb{P}(\varphi(i, U_i) = j)$ 

 $= P(i + 1_{(i-6)} = j)$ 

 $= \mathbb{P}(\underline{1}_{(0,-6)} = \delta^{-i})$ 

- 1(j=in) + - 1(j=i) Pij =