Negative binomial random variable (1)

Notation:

$$X \sim \text{Nbin}(r, p), \text{ for } r \in \mathbb{N}^*, \; p \in (0,1)$$

State space:

$$\{0, 1, 2 \ldots\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{k + r - 1}{k} p^r \, q^k, \quad k \geq 0$$

Expected value, variance and pgf:

$$\mathbf{E}[X] = \frac{r\,q}{p}, \qquad \mathbf{Var}(X) = \frac{r\,q}{p^2}, \qquad G_X(s) = \left(\frac{p}{1 - (1-p)s}\right)^r$$

Negative binomial random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ failures until r successes

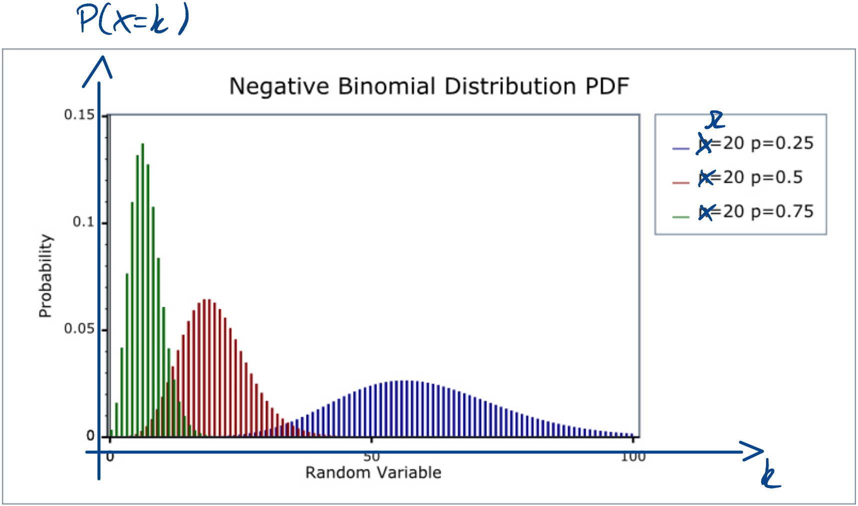
Justification:

$$(X = k) = ((r-1) \text{ successes in } (k+r-1) \text{ 1st trials}) \cap ((k+r)\text{-th trial is a success})$$

Thus

$$\mathbf{P}(X = k) = \binom{k-1}{r-1} p^r q^k$$

Negative binomial random variable (3)



Negative binomial random variable for $r = 1$

Rmk : We can write $X = Y - 1$ where $Y \sim \mathcal{G}(p)$

Notation:

$$X \sim \text{Nbin}(1, p), \text{ for } r \cancel{\in \mathbb{N}^*}, \; p \in (0,1)$$

State space:

$$\{0, 1, 2 \dots\}$$

Pmf:

$$\mathbf{P}(X = k) = p\, q^k, \quad k \geq 0$$

Expected value, variance and pgf:

$$\mathbf{E}[X] = \frac{q}{p}, \qquad \mathbf{Var}(X) = \frac{q}{p^2}, \qquad G_X(s) = \frac{p}{1 - (1-p)s}$$

$= Var(Y)$

Expression for G_X

$$\boxed{G_X(s) =} \sum_{k=0}^{\infty} P(X=k)\, s^k$$

$$= \sum_{k=0}^{\infty} p\, q^k s^k$$

$$= p \sum_{k=0}^{\infty} (qs)^k \qquad \text{(geom. series)}$$

$$\boxed{= \frac{p}{1-qs}}$$

Particular case

$p = \frac{1}{2}$, then

$$G_X(s) = \frac{1/2}{1 - s/2} = \frac{1}{2-s}$$

Branching with negative binomial offspring

$p=\frac{1}{2} \Rightarrow E[Z_1]=1$ $\quad$ $p>\frac{1}{2} \Rightarrow E[Z_1]>1$

$p<\frac{1}{2} \Rightarrow E[Z_1]<1$

Proposition 20.

For the branching process with $Z_1 \sim \text{Nbin}(1, p)$ we have

1. The generating function G_n is given by

$$G_n(s) = \begin{cases} \frac{n-(n-1)s}{n+1-ns} & \text{if } p = \frac{1}{2} \\ \frac{q(p^n-q^n)-ps\left(p^{n-1}-q^{n-1}\right)}{p^{n+1}-q^{n+1}-ps(p^n-q^n)} & \text{if } p \neq \frac{1}{2} \end{cases}$$

2. The probability of extinction is

$$\mathbf{P}(\text{Ultimate extinction}) = \begin{cases} 1 & \text{if } p \leq \frac{1}{2} \\ \frac{q}{p} & \text{if } p > \frac{1}{2} \end{cases}$$

Rmk We use $Z_1 \sim Nbin(1,p)$ because

(i) $Z_1 = 0, 1, \ldots$ → good for # offsprings

(ii) G_n is "easily" computed

It is not related to Bernoulli trials

Proof for $p=\frac{1}{2}$. Claim: $G_n(s) = \dfrac{n-(n-1)s}{n+1-ns}$

Initial step $G_{X_0}(s) = \sum_{k=0}^{\infty} s^k P(X_0=k) = s$

$(P(X_0=1)=1)$

and $G_0(s) = \dfrac{0+s}{1} = s$

Induction If claim is true for n, then

$\boxed{G_{Z_{n+1}}(s) =}$ $G(G_n(s))$ $\quad \left(G(x) = \dfrac{1}{2-x}\right)$

$$= \frac{1}{2 - \frac{n-(n-1)s}{n+1-ns}}$$

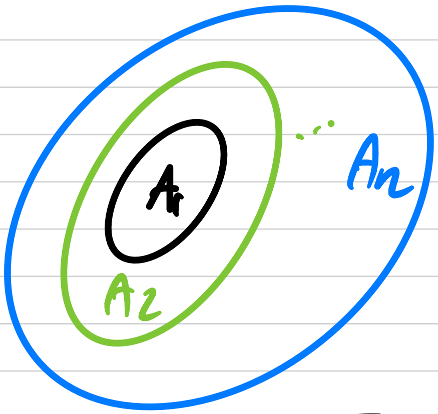
$$= \frac{n+1-ns}{2n+2-2ns-n+(n-1)s} = \frac{n+1-ns}{n+2-(n+1)s}$$

$$\boxed{= G_{n+1}(s)}$$

Proof for ultimate extinction

Recall: Consider a sequence $\{A_n ; n \geq 1\}$ which is increasing ($A_n \subset A_{n+1}$). Then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$$



Application Consider

$$A_n = (Z_n = 0)$$

Then (i) $Z_n = 0 \Rightarrow Z_{n+1} = 0$

Thus $A_n = (Z_n = 0) \subset A_{n+1} = (Z_{n+1} = 0)$

(ii) $\boxed{P(\text{ultimate extinction}) = P\left(\bigcup_{n=0}^{\infty} (Z_n = 0)\right)}$

$$= P\left(\bigcup_{n=0}^{\infty} A_n\right)$$

$$G_n(s) = \frac{n-(n-1)s}{n+1-ns}$$

Summary We have seen

$$P(\text{ultimate extinction}) = \lim_{n\to\infty} P(Z_n = 0)$$

In terms of G_n

$$G_n(s) = \sum_{k=0}^{\infty} P(Z_n = k)\, s^k$$

$$P(Z_n = 0) = G_n(0)$$

$$= \frac{n}{n+1}$$

$$\lim_{n\to\infty} P(Z_n = 0) = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Thus $\boxed{P(\text{extinction}) = 1 \quad \text{if} \quad p = \frac{1}{2}}$

Proof of Proposition 20 (1)

Pgf for Z_1: Since $Z_1 \sim \text{Nbin}(1, p)$ we have

$$G_{Z_1}(s) = \frac{p}{1-(1-p)s}$$

Expression for G_n: One can check that

$$G\left(G_n(s)\right) = G_{n+1}(s)$$

Proof of Proposition 20 (2)

Ultimate extinction: We set

$$A = (\text{Ultimate extinction occurs})$$

Then

$$A = \bigcup_{n \geq 1} A_n, \quad \text{with} \quad A_n = (Z_n = 0)$$

$\mathbf{P}(A)$ as a limit: We have

$$A_n \subset A_{n+1} \quad \Longrightarrow \quad \mathbf{P}(A) = \lim_{n \to \infty} \mathbf{P}(A_n)$$

Proof of Proposition 20 (3)

Expression for $\mathbf{P}(A_n)$: We have

$$\mathbf{P}(A_n) = G_n(0) = \begin{cases} \frac{n}{n+1} & \text{if } p = \frac{1}{2} \\ \frac{q(p^n - q^n)}{p^{n+1} - q^{n+1}} & \text{if } p \neq \frac{1}{2} \end{cases}$$

Expression for $\mathbf{P}(A)$: We obtain

$$\mathbf{P}(A) = \lim_{n \to \infty} \mathbf{P}(A_n) = \begin{cases} 1 & \text{if } p \leq \frac{1}{2} \\ \frac{q}{p} & \text{if } p > \frac{1}{2} \end{cases}$$