Ultimate extinction in the general case

Theorem 21.

average # offspring = 1 ⇒ extinction

Consider a branching process with

- $Z_1 \sim f$, f with pgf G
- $\mu = \mathbf{E}[Z_1]$ and $\sigma^2 = \mathbf{Var}(Z_1)$

Let

- $\eta \equiv$ smallest non-negative root of $s = G(s)$

Then

1. $\mathbf{P}(\text{Ultimate extinction}) = \eta$
2. $\eta = 1$ if $\mu < 1$ (average # offspring < 1 ⇒ extinction)
3. $\eta < 1$ if $\mu > 1$ (average # offspring > 1 ⇒ > 0 prob to survive)
4. $\eta = 1$ if $\mu = 1$ and $\sigma^2 > 0$

Recall we have seen

$$P(\text{extinction}) = \lim_{n\to\infty} P(A_n),$$

where $A_n = (Z_n = 0)$

In addition

$$P(A_n) = P(Z_n = 0) = G_n(0) = G^{\circ(n)}(0)$$

Iterated sequence

Call $\eta_n = P(Z_n = 0)$. Then

(i) $\eta_0 = P(Z_0 = 0) = 0$

(ii) $\eta_{n+1} = G(\eta_n)$

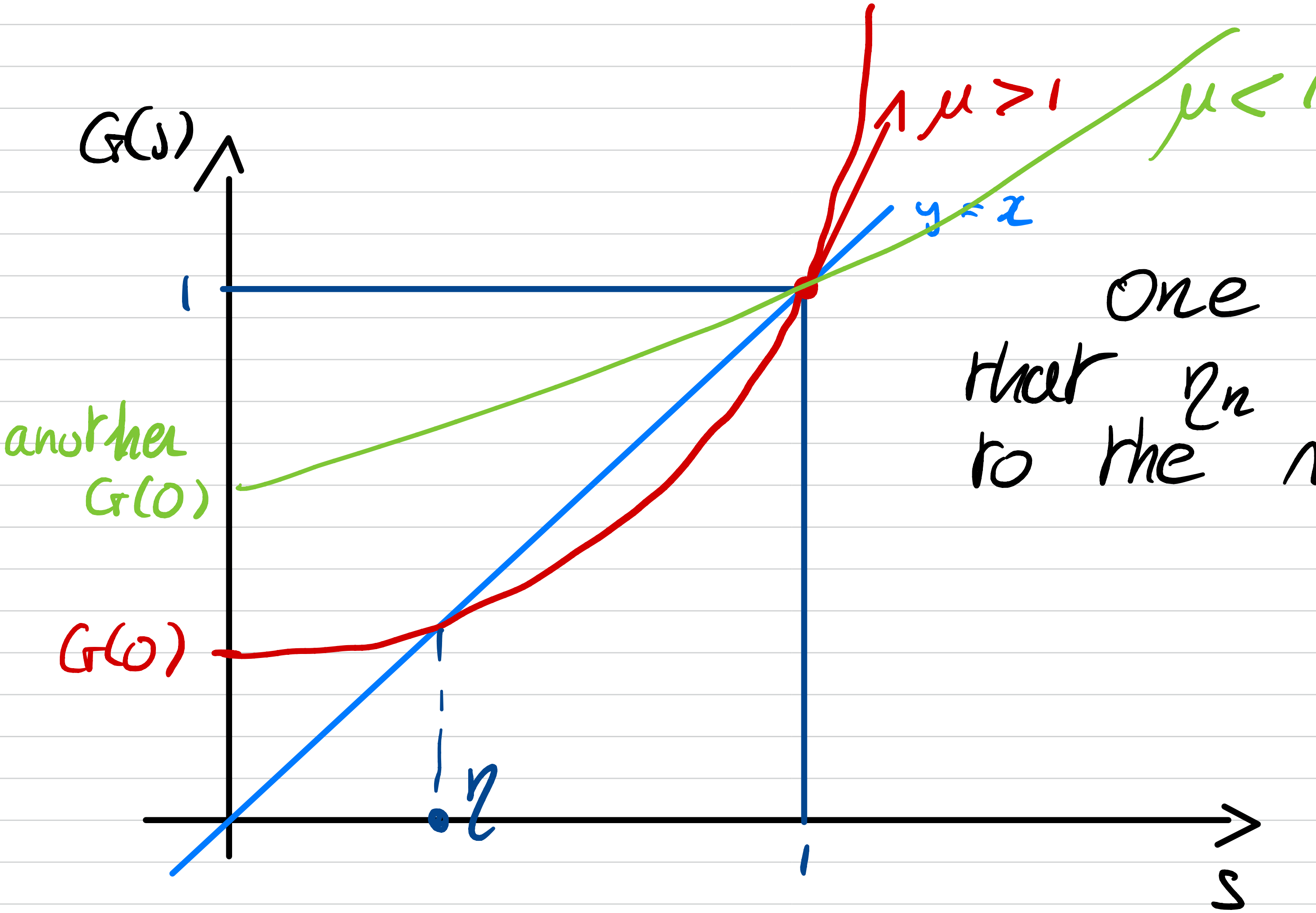
Rmk If a sequence $x_{n+1} = G(x_n)$ converges to x, then

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} G(x_n)$$

$$x = G(x)$$

Thus the limit is a root of

$$s = G(s)$$



One can prove that η_n is $\nearrow$ to the root $\eta = G(\eta)$

$$G(0) = P(Z_1 = 0) \in (0,1)$$

$$G(1) = \sum_{k=0}^{\infty} P(Z_1 = k) = 1$$

$$G'(1) = E[Z_1] = \mu$$

Rmk One can prove this type of convergence if G is

(i) Increasing

(ii) Convex

Here (i) $G(s) = E[s^{X_1}]$, with $X_1 \geq 0$

$\Rightarrow$ $s \mapsto s^{X_1}$ increasing

$\Rightarrow$ $s \mapsto E[s^{X_1}]$ increasing

Also $\boxed{G'(s)} = E[X_1 \, s^{X_1 - 1}] \boxed{\geq 0}$

(ii) $G''(s) = E[X_1 (X_1 - 1) \, s^{X_1 - 2}] \geq 0$

$\Rightarrow$ $\boxed{G \text{ convex}}$

Proof of Theorem 21 (1)

Ultimate extinction: Recall that we have set

$$A = (\text{Ultimate extinction occurs})$$

Then

$$A = \bigcup_{n \geq 1} A_n, \quad \text{with} \quad A_n = (Z_n = 0)$$

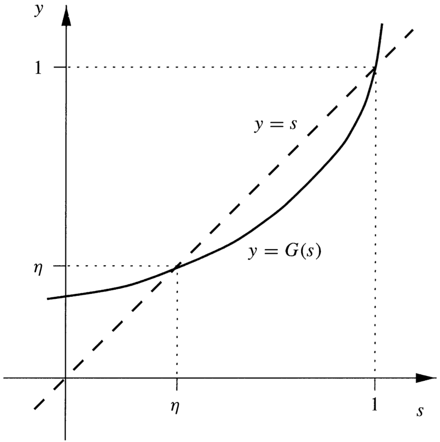
$\mathbf{P}(A)$ as a limit: We have $A_n \subset A_{n+1}$. Thus

$$\eta_n \equiv \mathbf{P}(A_n) \text{ is } \nearrow, \quad \text{and} \quad \mathbf{P}(A) = \lim_{n \to \infty} \eta_n$$

Proof of Theorem 21 (2)

Claim when $\mu > 1$:

$$G(0) \in [0,1)\ ,\ G'(0) \in [0,1)\ ,\ G'(1) > 1\ ,\ G \text{ convex on } [0,1]$$



Proof of Theorem 21 (3)

Claim $G(0) \in [0,1)$: We have

$$G(0) = \mathbf{P}(Z_1 = 0) < 1 \quad \text{(otherwise trivial extinction)}$$

Claim $G'(0) \in [0,1)$: Write

$$G'(0) = \mathbf{P}(Z_1 = 1) < 1 \quad \text{(or trivial offspring} = 1\text{)}$$

Claim $G'(1) > 1$: One argues

$$G'(1) = \mu > 1$$

Claim G convex on $[0,1]$: We compute

$$G''(s) = \mathbf{E}\left[Z_1(Z_1 - 1)s^{Z_1 - 2}\right] \geq 0$$

Proof of Theorem 21 (4)

Conclusion: Follows classical lines for sequences

$$\eta_{n+1} = G(\eta_n) \quad \Longrightarrow \quad \lim_{n\to\infty} \eta_n = \eta$$