Outline

Vocabulary

random walks X_n
branching process Z_n

Stochastic process:

- Family $\{X_n;\ n \geq 0, n \text{ integer}\}$ of random variables
- Family evolving in a random but prescribed manner
- Here $X_n \in S$, where S countable state space with $N = |S|$

S countable : there is a one-to-one correspondance between elements of S and $\mathbb{N}$ or

Discrete time:

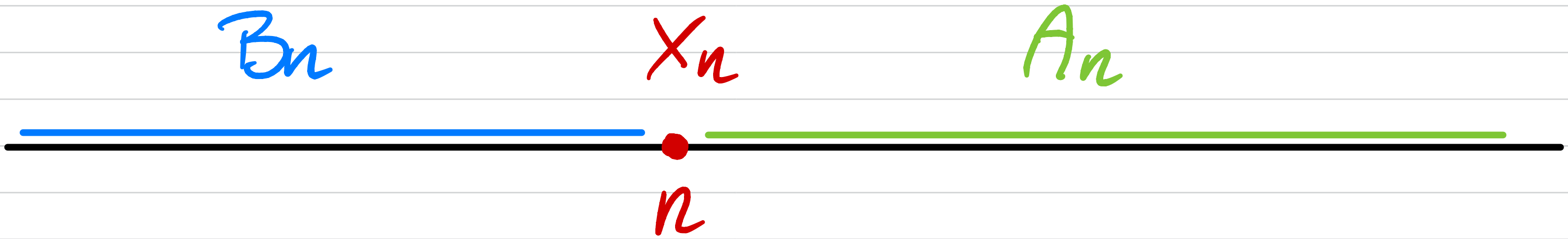
- In this chapter we consider X indexed by $n \in \mathbb{N}$, discrete any $E \subset \mathbb{N}$
- Later continuous time, $\{X_t;\ t \geq 0\}$

Ex of countable sets:
$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$

Ex of non countable set:
$\mathbb{R}, \mathbb{C}, \mathbb{R}^n, \mathbb{C}^n$

Markov evolution:

Conditioned on X_n,
the evolution does not depend on the past



Conditioned on X_n, $A_n \perp\!\!\!\perp B_n$

Markov chain

conditioning on the past is the same as conditioning on the present

Definition 1.

Let

- $X = \{X_n;\ n \geq 0, n \text{ integer}\}$ stochastic process

We say that X is a Markov chain if

$$\mathbf{P}\left(X_n = s \middle| X_0 = x_0, \ldots, X_{n-1} = x_{n-1}\right) = \mathbf{P}\left(X_n = s \middle| X_{n-1} = x_{n-1}\right),$$

for all $n \geq 1$ and $x_0, \ldots, x_{n-1}, s \in S$

Random walk as a Markov chain

Proposition 2.

Let

- $X_1, \ldots, X_n$ Bernoulli random variables with values ± 1,

$$\mathbf{P}(X_i = 1) = p, \qquad \mathbf{P}(X_i = -1) = 1 - p$$

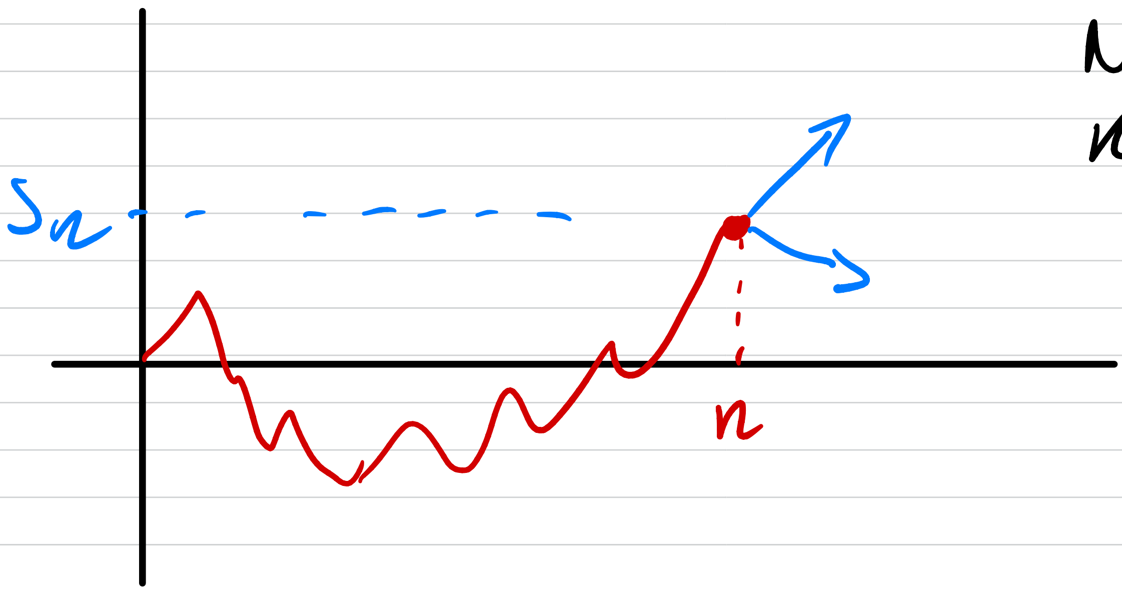
- The X_i's are independent
- The random walk defined by $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} X_i$$

Then

S is a Markov chain

Intuitively



Next move after n is

$S_n \pm 1$

Proof that S_n is a Markov chain. Recall

$$S_{n+1} = S_n + X_{n+1}$$

$x_0, \dots, x_n \in \mathbb{Z}$

Thus

$$P(S_{n+1} = s \mid S_0 = x_0, \dots, S_n = x_n)$$

$$= P(S_n + X_{n+1} = s \mid S_0 = x_0, \dots, S_n = x_n)$$

$$= P(X_{n+1} = s - x_n \mid S_0 = x_0, \dots, S_n = x_n)$$

Note
- $X_{n+1} \perp\!\!\!\perp (S_0, S_1, \dots, S_n)$
- $P(A \mid B) = P(A)$ if $A \perp\!\!\!\perp B$

$$= P(X_{n+1} = s - x_n)$$

$$= P(S_{n+1} = s \mid S_n = x_n) \Rightarrow \boxed{S \text{ Markov chain}}$$

Proof of Proposition 2

Decomposition for S_n: We write

$$S_{n+1} = S_n + X_{n+1}$$

Conditional probability: We have

$$\begin{aligned}
&\mathbf{P}\left(S_{n+1} = s \middle| S_0 = x_0, \ldots, S_n = x_n\right) \\
&= \mathbf{P}\left(S_n + X_{n+1} = s \middle| S_0 = x_0, \ldots, S_n = x_n\right) \\
&= \mathbf{P}\left(X_{n+1} = s - x_n \middle| S_0 = x_0, \ldots, S_n = x_n\right) \\
&= \mathbf{P}\left(X_{n+1} = s - x_n\right) \\
&= \mathbf{P}\left(S_{n+1} = s \middle| S_n = x_n\right)
\end{aligned}$$

This proves the Markov property

Alternative formulations for Markov's property

X random variable
x numerical value

Proposition 3.

The Markov property is equivalent to any of the following:

1. For all $n_1 < n_2 < \cdots < n_k \leq n$ we have

   conditioning on a part of the past

   $$\mathbf{P}\left(X_n = s \middle| \overbrace{X_{n_1} = x_{n_1}, \ldots, X_{n_k} = x_{n_k}}\right) = \mathbf{P}\left(X_n = s \middle| X_{n_k} = x_{n_k}\right)$$

2. $n > 0$

   For all $m, n \geq 0$,

   conditioning on the remote past

   $$\mathbf{P}\left(X_{m+n} = s \middle| \overbrace{X_0 = x_0, \ldots, X_m = x_m}\right) = \mathbf{P}\left(X_{m+n} = s \middle| X_m = x_m\right)$$

Transition probability

Reduction to $S \subset \mathbb{N}$: $\{\omega \in \Omega ;\ X_n(\omega) = x_i\}$

- Recall that $X_n \in S$
- S countable $\Longrightarrow$ S in one-to-one correspondence with $S' \subset \mathbb{N}$
- We denote $(X_n = x_i)$ by $(X_n = i)$

Important quantity to describe X: Transition probability, defined by

$$\mathbf{P}\left(X_{n+1} = j \mid X_n = i\right) = p_{ij}(n)$$

It depends on n, i, j

Andrey Markov

Markov chains are still used for speech recognition

Andrey Markov's life:

- Lifespan: 1856-1922, $\simeq$ St Petersburg
- Not a very good student
  $\hookrightarrow$ except in math
- Contributions in analysis and probability
- Used chains for
  $\hookrightarrow$ appearance of vowels
- Professor in St Petersburg
  - Suspended after 1908 students riots
  - Resumed teaching in 1917

Fact: More than 50 mathematical objects named after Markov!!

Homogeneous Markov chains

General case

$P(X_{n+1}=j \mid X_n=i) = p_{ij}(\cancel{n})$

homogeneous

Definition 4.

Let X be a Markov chain. Then

1. X is homogeneous if for all n, i, j we have

$$\mathbf{P}\left(X_{n+1}=j \mid X_n=i\right)=\mathbf{P}\left(X_1=j \mid X_0=i\right)$$

2. If X is homogeneous we define a transition matrix

$$P=\left(p_{ij}\right) \quad \text{with} \quad p_{ij}=\mathbf{P}\left(X_{n+1}=j \mid X_n=i\right)$$

Hypothesis 5.

In the chapter we always assume that X is homogeneous

Stochastic matrix

Theorem 6.

The matrix P is stochastic, that is

1. $p_{ij} \geq 0$, for all i, j
2. $\sum_j p_{ij} = 1$, for all i

<u>Recall</u> $\quad p_{ij} = P(X_{n+1} = j \mid X_n = i)$

Then

(i) $\quad p_{ij}$ is a (conditional) probability

$\Rightarrow \quad p_{ij} \in [0,1]$

In particular, $\boxed{p_{ij} \geq 0}$

(ii) $\quad \boxed{\sum_{j \in S} p_{ij}} = \sum_{j} P(\overbrace{X_{n+1} = j}^{\text{disjoint events}} \mid X_n = i)$

$= P\left(\bigcup_{j \in S} (X_{n+1} = j) \mid X_n = i \right)$

If $P(A) = 1$, then $P(A \cap C) = P(C)$

$= P(X_{n+1} \in S \mid X_n = i)$

$= \dfrac{P((X_{n+1} \in S) \cap X_n = i)}{P(X_n = i)} = \dfrac{P(X_n = i)}{P(X = i)} \boxed{= 1}$