Easy criteria to establish Markov property

Proposition 10.

Let X be a process such that

- $X_{n+1} = \varphi(X_n, Z_{n+1})$ $\longrightarrow$ pathwise description
- $Z_{n+1} \perp\!\!\!\perp (X_0, \ldots, X_n)$
- $\{Z_n;\, n \geq 1\}$ i.i.d family
- φ is a given fixed function

Then

homogeneous

1. X is a Markov chain
2. The transition is given by

average description

$$p_{ij} = \mathbf{P}\left(\varphi(i, Z_1) = j\right)$$

Prop $X_{n+1} = \varphi(X_n, Z_{n+1}) \Rightarrow X$ Markov chain

Rmk The Prop says that if

X_{n+1} = function of X_n & an innovation Z_{n+1}

$\Rightarrow X_{n+1}$ Markov

Rmk 2 X_{n+m} of the form

$$X_{n+m} = F_m(X_n, Z_{n+1}, \dots, Z_{n+m})$$

Ex: $X_{n+2} = \varphi(X_{n+1}, Z_{n+2})$

$$= \varphi(\varphi(X_n, Z_{n+1}), Z_{n+2})$$

$$\equiv F_2(X_n, Z_{n+1}, Z_{n+2})$$

Prop $X_{n+1} = \varphi(X_n, Z_{n+1}) \Rightarrow X$ Markov chain

Rmk 3 Proof: as an exercise.
very similar to the proof of random walk is a Markov chain

Rmk 4 The implication

$$X_{n+1} = \varphi(X_n, Z_{n+1}) \Leftarrow X \text{ Markov chain}$$

is also true.

This is much harder to prove

Application 1 : $X_n = \sum_{k=1}^{n} Z_k$ random walk is a Markov chain

Decomposition

$X_{n+1} = X_n + Z_{n+1} = \varphi(X_n, Z_{n+1})$

with

$\varphi(x, z) = x + z$

Z_k iid with $P(Z_k = 1) = p$, $P(Z_k = -1) = q$

Conclusion $\boxed{X \text{ is a Markov chain}}$

<u>Transition</u> According to Prop 4,

$$p_{ij} = P(\varphi(i, z_1) = j)$$

$$= P(i + z_1 = j)$$

$$p + q = 1$$

$$= P(z_1 = j - i)$$

$$P = (p_{ij})_{\substack{i \in \mathbb{Z} \\ j \in \mathbb{Z}}}$$

$$= \begin{cases} p & \text{if } j - i = 1 \\ q & \text{if } j - i = -1 \\ 0 & \text{otherwise} \end{cases}$$

$$= P$$

$$\begin{pmatrix} q & 0 & p & & & & 0 \\ & \ddots & \ddots & \ddots & & & \\ & & q & 0 & p & & \\ & & & q & 0 & p & \\ 0 & & & & \ddots & \ddots & \ddots \\ & & & & q & 0 & p \end{pmatrix}$$

$$= \begin{cases} p & \text{if } j = i+1 \\ q & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

$$P = \begin{pmatrix} q & 0 & p & & & & \\ & q & 0 & p & & & \\ & & q & 0 & p & & \\ & 0 & & \ddots & \ddots & \ddots & \\ & & & & q & 0 & p \end{pmatrix} \leftarrow i$$

q p q p

$$\Rightarrow \quad p_{11} = 0, \quad p_{12} = p, \quad p_{10} = q$$

$$P^2 = \begin{pmatrix} q^2 & 0 & \cdots & 0 & p^2 & & 0 \\ & \ddots & & & & \ddots & \\ & & q^2 & 0 & 2pq & 0 & p^2 \\ & 0 & & \ddots & & \ddots & 0 \\ & & & q^2 & 0 & & 2pq \end{pmatrix}$$

Distribution of X_n

If $X_0 = i$ then

$$X_n = i + \sum_{k=1}^{n} Z_k$$

$Z_k = 2Y_k - 1$ with $Y_k \sim B(p)$

$$= i + \sum_{k=1}^{n} (2Y_k - 1)$$

$$= i + 2\sum_{k=1}^{n} Y_k - n$$

Thus

$$P_{ij}(n) = P(X_n = j \mid X_0 = i)$$

$\sum_{k=1}^{n} Y_k \sim Bin(n, p)$

$$= P\left(i + 2\sum_{k=1}^{n} Y_k - n = j\right)$$

$$= P\left(\sum_{k=1}^{n} Y_k = \frac{1}{2}(n + j - i)\right)$$

- $i - n \leq j \leq i + n$

$$= \binom{n}{\frac{1}{2}(n+j-i)} p^{\frac{1}{2}(n+j-i)} q^{\frac{1}{2}(n+i-j)}$$

- $j - i$ has same parity as n