Branching process example

$S = \mathbb{N}$

Dynamics: $X_{n+1} = \sum_{k=1}^{X_n} Z_k^{(n+1)} = \sum_{k=1}^{\infty} Z_k \mathbb{1}(k \le X_n)$

Thus

$$X_{n+1} = \varphi(X_n, Z^{(n+1)})$$

where $Z^{(n+1)} = \{ Z_k^{(n+1)} ; k \geq 1 \}$, $\overbrace{Z_k^{(n+1)}}^{\text{generation}}_{\text{offspring } \# k} \in \mathbb{N}$

$$\varphi(x, \overbrace{Z}^{\text{sequence}}) = \sum_{k=1}^{\infty} Z_k \mathbb{1}(k \le x)$$

The sequences are i.i.d

$\Rightarrow$ $\boxed{X \text{ is a Markov chain}}$

$$\varphi(i,z) = \sum_{k=1}^{i} z_k = \sum_{k=1}^{\infty} z_k \mathbb{1}_{(k \le i)}$$

Transition

$$P_{ij} = P(\varphi(i, z^{(1)}) = j)$$

$$= P\left(\sum_{k=1}^{i} z_k^{(1)} = j \right)$$

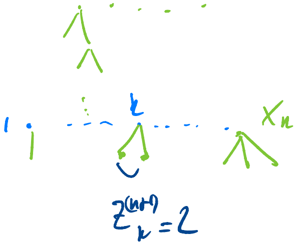
We have seen that those quantities could be expressed in terms of generating functions. If $G \equiv$ pgf for $z_1^{(1)}$, then

$$\mathrm{pgf}\left(\sum_{k=1}^{i} z_k^{(1)} \right) = G^i$$

Thus

$$P\left(\sum_{k=1}^{i} z_k^{(1)} = j \right) = \frac{1}{j!} \left\{ \text{coefficient of } s^j \text{ in the function } [G(s)]^i \right\}$$

Branching process case (1)



State space:

$$S = \mathbb{N}$$

Markov property: We have seen

- $X_{n+1} = \sum_{k=1}^{X_n} Z_k^{(n+1)} = \varphi(X_n, \mathbf{Z}^{(n+1)})$
- $\mathbf{Z}^{(n)} = \{\mathbf{Z}_k^{(n)};\ k \geq 1\}$ is a sequence
- $\varphi(x, \mathbf{z}) = \sum_{k=1}^{x} z_k$
- $\{\mathbf{Z}^{(n)};\ n \geq 1\}$ i.i.d family
  $\hookrightarrow$ with $(Z_k^{(n)})_{k \geq 1}$ i.i.d with common pgf G

Thus

X is a Markov chain

Branching process case (2)

Transition probability: We have

$$\begin{aligned} p_{ij} &= \mathbf{P}\left(\sum_{k=1}^{i} Z_k^{(1)} = j\right) \\ &= \frac{1}{j!} \times \text{Coefficient of } s^j \text{ in } (G(s))^i \end{aligned}$$

n-step transition: We obtain

$= (G^{\circ n}(s))^i$

$$p_{ij}(n) = \frac{1}{j!} \times \text{Coefficient of } s^j \text{ in } (G_n(s))^i$$

Outline

Questions about Markov chains

Main questions

1. Does the MC X_n go to ∞ when $n \to \infty$?
2. Does it return to state i after $n = 0$?
3. How often does it return to i?
4. What is the range of $X_n(\omega)$?

Methodologies to answer those questions

1. We have seen: pgf's for random walks and branching
2. Now: Markov chain methods

Persistent and transient states

Definition 11.

Let

- X Markov chain
- i state in S

Then

1. i is called persistent or recurrent if

   we will always return to i

$$\mathbf{P}\left(X_n = i \text{ for some } n \geq 1 \middle| X_0 = i\right) = 1$$

2. i is called transient if

   we are not sure to return to i

$$\mathbf{P}\left(X_n = i \text{ for some } n \geq 1 \middle| X_0 = i\right) < 1$$

First passage time probabilities

Definition 12.

Let

- X Markov chain and i, j states in S

Then we define

1. Probability that
   $\hookrightarrow$ 1st visit to j starting from i takes place at step n:

$$f_{ij}(n) = \mathbf{P}\left(X_1 \neq j, \ldots, X_{n-1} \neq j, X_n = j \middle| X_0 = i\right)$$

2. Probability that X ever visits j starting from i:

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n)$$

Recall that $f_{ij}(n) = P(X_1 \neq j, \dots, X_{n-1} \neq j, X_n = j \mid X_0 = i)$

Set

$$T_j = \inf\{k \geq 1 ; X_k = j\}$$

Then (i) $f_{ij}(n) = P(T_j = n \mid X_0 = i)$

(ii) $f_{ij} = \sum_{n=1}^{\infty} f_{ij}(n) =$

$$= \sum_{n=1}^{\infty} P(T_j = n \mid X_0 = i)$$

$$= P(T_j < \infty \mid X_0 = i)$$

If $\boxed{f_{ii} = 1,}$ then $P(T_i < \infty \mid X_0 = i) = 1$

$\boxed{\Rightarrow \ i \ \text{persistent}}$

Alternative definition for $f_{ij}(n)$

First visit to j: We set $T_j = \infty$ if there is no visit to j, and

$$T_j = \inf\{n \geq 1;\, X_n = j\}$$

Expression for $f_{ij}(n)$: We have

$$\begin{aligned} f_{ij}(n) &= \mathbf{P}\left(X_1 \neq j, \ldots, X_{n-1} \neq j, X_n = j \,\middle|\, X_0 = i\right) \\ &= \mathbf{P}\left(T_j = n \,\middle|\, X_0 = i\right) \end{aligned}$$

Some pgf's

Pgf's P and F: We set

$$\overset{\text{generating function}}{P_{ij}(s)} = \sum_{n=0}^{\infty} p_{ij}(n)s^n , \qquad \overbrace{F_{ij}(s) = \sum_{n=0}^{\infty} \underbrace{f_{ij}(n)}_{P(T_j = n \mid X_0 = i)} s^n}^{\text{pgf for } T_j \text{ conditioned on } (X_0 = i)}$$

$$p_{ij}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Remarks:

1. Conventions above: $p_{ij}(0) = \delta_{ij}$ and $f_{ij}(0) = 0$
2. i persistent iff $f_{ii} = 1$
3. For $|s| < 1$, the series $P_{ij}(s)$ and $F_{ij}(s)$ are convergent
4. $P_{ij}(1)$ and $F_{ij}(1)$ are defined through Abel's theorem
5. $f_{ij} = F_{ij}(1)$

Relation between F and P

Theorem 13.

Let X_n be a Markov chain with transition p. Then

1. P_{ii} and F_{ii} satisfy

$$P_{ii}(s) = 1 + F_{ii}(s)P_{ii}(s)$$

2. For $i \neq j$, the function P_{ij} verifies

$$P_{ij}(s) = F_{ij}(s)P_{jj}(s)$$