Relation between F and P

Theorem 13.

Let X_n be a Markov chain with transition p. Then

1. P_{ii} and F_{ii} satisfy

$$P_{ii}(s) = 1 + F_{ii}(s)P_{ii}(s)$$

2. For $i \neq j$, the function P_{ij} verifies

$$P_{ij}(s) = F_{ij}(s)P_{jj}(s)$$

<u>Strategy for proof</u>: Get a type of convolution.

<u>Computation</u> start from $P(X_n = j \mid X_0 = i)$

$$\boxed{P(X_n = j \mid X_0 = i)} = P(X_n = j, T_j \leq n \mid X_0 = i)$$

$$= \sum_{k=1}^{n} P(X_n = j, T_j = k \mid X_0 = i) \quad \left(\text{disjoint } \bigcup_{k=1}^{n} (T_j = k)\right)$$

<u>Recall</u> $P(A \cap B \mid C) = P(A \mid B \cap C)\, P(B \mid C)$

$$= \sum_{k=1}^{n} P(X_n = j \mid T_j = k, X_0 = i)\, P(T_j = k \mid X_0 = i)$$

$X_1 \neq j, X_2 \neq j, \dots, X_{k-1} \neq j, X_k = j$

Markov
$$= \sum_{k=1}^{n} P(X_n = j \mid X_k = j)\, f_{ij}(k)$$

$$\boxed{= \sum_{k=1}^{n} P_{jj}(n-k)\, f_{ij}(k)}$$

convolution type relation

Proof of Theorem 13 (1)

Events: We set

- $$A_m = (X_m = j), \qquad B_k = (T_j = k)$$

Decomposition for A_m: We have

$$A_m = A_m \cap \left(\bigcup_{k=1}^{n} B_k \right) = \bigcup_{k=1}^{n} (A_m \cap B_k)$$

Proof of Theorem 13 (2)

Preliminary identity: Recall that

$$\mathbf{P}(A \cap B|\, C) = \mathbf{P}(A|\, B \cap C)\mathbf{P}(B|\, C)$$

Decomposition for probabilities: We get

$$\begin{aligned} \mathbf{P}(A_m \cap B_k|\, X_0 = i) &= \mathbf{P}\left(A_m|\, B_k,\, X_0 = i\right) \mathbf{P}(B_k|\, X_0 = i) \\ &\overset{\text{Markov}}{=} \mathbf{P}\left(A_m|\, X_k = j\right) \mathbf{P}(B_k|\, X_0 = i) \qquad (1) \end{aligned}$$

Proof of Theorem 13 (3)

Convolution relation: Equation (1) can be read as

$$
\begin{aligned}
p_{ij}(m) &= \mathbf{P}(A_m |\, X_0 = i) \\
&= \sum_{k=1}^{n} \mathbf{P}(A_m \cap B_k |\, X_0 = i) \\
&= \sum_{k=1}^{n} p_{jj}(m-k) f_{ij}(k), \quad \text{for} \quad m \geq 1, \quad \text{and} \quad p_{ij}(0) = \delta_{ij}
\end{aligned}
$$

Expression with generating functions: We get

$$P_{ij}(s) - \delta_{ij} = F_{ij}(s) P_{jj}(s)$$

Criterion for recurrence and transience

Potential of the Markov chain
U_j

Proposition 14.

Let X_n be a Markov chain with transition p. Then

1. If $\sum_{n=0}^{\infty} p_{jj}(n) = \infty$, then
   - State j is persistent
   - $\sum_{n=0}^{\infty} p_{ij}(n) = \infty$ for all i's such that $f_{ij} > 0$

2. If $\sum_{n=0}^{\infty} p_{jj}(n) < \infty$, then
   - State j is transient
   - $\sum_{n=0}^{\infty} p_{ij}(n) < \infty$ for all i

rw in $\mathbb{Z}$ is recurrent
$\mathbb{Z}^2$ " "
$\mathbb{Z}^3$ " transient

Proof We have seen $P_{jj}(s) = 1 + F_{jj}(s)\, P_{jj}(s)$

Thus

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)} \quad \text{if} \quad |s| < 1$$

Take $s \to 1$. We get that

$$\lim_{s \to 1} P_{jj}(s) = \infty \quad \text{iff} \quad F_{jj}(1) = 1$$

Now

- $\lim_{s \to \infty} P_{jj}(s) = \sum_{n=0}^{\infty} p_{jj}(n) = U_j$
- $F_{jj}(1) = f_{jj} = \sum_{k=1}^{\infty} P(T_j = k \mid X_0 = j)$
- $f_{jj} = 1 \iff j$ persistent

Thus $\boxed{U_j = \infty \text{ iff } j \text{ persistent}}$

Proof of Proposition 14 (1)

Expression for $P_{jj}(s)$: From Theorem 13 we have

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)}, \quad \text{for} \quad |s| < 1$$

Limit as $s \nearrow 1$: We get

- $P_{jj}(s) \to \infty$ iff $F_{jj}(1) = 1$
- $F_{jj}(1) = f_{jj}$
- j persistent iff $f_{jj} = 1$

Thus

$$j \text{ persistent iff } \lim_{s \nearrow 1} P_{jj}(s) = \infty$$

Proof of Proposition 14 (2)

Recall: We have seen

$$j \text{ persistent iff } \lim_{s \nearrow 1} P_{jj}(s) = \infty$$

Application of Abel:

$$\lim_{s \nearrow 1} P_{jj}(s) = \sum_{n=0}^{\infty} p_{ij}(n)$$

Conclusion:

$$j \text{ persistent iff } \sum_{n=0}^{\infty} p_{ij}(n) = \infty$$

Proof of Proposition 14 (3)

Another relation for $p_{ij}(n)$: We have seen

$$P_{ij}(s) = F_{ij}(s) P_{jj}(s)$$

Taking limits $s \nearrow 1$ we get

$$\sum_{n=0}^{\infty} p_{ij}(n) = f_{ij} \sum_{n=0}^{\infty} p_{jj}(n)$$

Conclusion: If $\sum_{n=0}^{\infty} p_{jj}(n) = \infty$, then

$\sum_{n=0}^{\infty} p_{ij}(n) = \infty$ for all i's such that $f_{ij} > 0$

Behavior of $p_{ij}(n)$

Proposition 15.

Let

- X Markov chain with transition p
- j transient state

Then

$$\lim_{n\to\infty} p_{ij}(n) = 0$$

Simple random walk case

Proposition 16.

Let

- X simple random walk
- Parameters p and $q = 1 - p$

Then

$$X \text{ is persistent iff } p = \frac{1}{2}$$

Rmk We have already proved that, by computing $F_{00}(s)$

Proof We want to apply the criterion

j persistent iff $\sum_{n=0}^{\infty} p_{jj}(n) = \infty$

Transition For m odd $p_{jj}(m) = 0$ and

$$p_{jj}(2n) = \binom{2n}{n} (pq)^n = \frac{(2n)!}{(n!)^2} (pq)^n$$

Next step Find $p_{jj}(2n) \overset{n\to\infty}{\sim}$?

Stirling formula: $n! \sim n^n e^{-n} \sqrt{2\pi n}$

Thus $$\frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi}\sqrt{2n}\,(2n)^{2n}\,e^{-2n}}{2\pi \; n \; n^{2n} \; e^{-2n}}$$

$$\sim \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{n}} 4^n$$

Equivalent

$$p_{jj}(2n) = \binom{2n}{n}(pq)^n \sim \frac{4^n}{\sqrt{\pi n}}(pq)^n$$

$$p_{jj}(2n) \sim \frac{1}{\sqrt{\pi}} \frac{1}{n^{\frac{1}{2}}} (4pq)^n$$

Recall: $4pq < 1$ for $p \neq \frac{1}{2}$

$c_p \equiv 4pq = 1$ for $p = \frac{1}{2}$

Case $p = \frac{1}{2}$

$$p_{jj}(2n) \sim \frac{1}{\sqrt{\pi}} \frac{1}{n^{\frac{1}{2}}}$$

$$\Rightarrow \boxed{\sum_n p_{jj}(2n) = \infty}$$

→ j persistent $\forall j$

Case $p \neq \frac{1}{2}$

$$p_{jj}(2n) \sim \frac{1}{\sqrt{\pi}} \frac{(c_p)^n}{n^{\frac{1}{2}}} \Rightarrow \boxed{\sum_n p_{jj}(2n) < \infty}$$

→ j transient $\forall j$

Proof of Proposition 16 (1)

Formula for $p_{jj}(m)$: According to (26),

$$p_{jj}(2n) = \binom{2n}{n}(pq)^n, \qquad p_{jj}(2n+1) = 0$$

Stirling's formula:

$$m! \equiv \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}$$

Equivalent for $p_{jj}(2n)$: We get, as $n \to \infty$,

$$p_{jj}(2n) \sim \frac{(4pq)^n}{(\pi n)^{1/2}}$$

Proof of Proposition 16 (2)

Recall: We have seen that

$$p_{jj}(2n) \sim \frac{(4pq)^n}{(\pi n)^{1/2}}$$

Case $p = \frac{1}{2}$: We get

$$p_{jj}(2n) \sim \frac{1}{(\pi n)^{1/2}}$$

Thus

$$\sum_{n=0}^{\infty} p_{jj}(2n) = \infty \quad \Longrightarrow \quad \text{State } j \text{ persistent}$$

Proof of Proposition 16 (3)

Recall: We have seen that

$$p_{jj}(2n) \sim \frac{(4pq)^n}{(\pi n)^{1/2}}$$

Case $p \neq \frac{1}{2}$: We get

$$p_{jj}(2n) \sim \frac{c_p}{(\pi n)^{1/2}}, \quad \text{with} \quad c_p < 1$$

Thus

$$\sum_{n=0}^{\infty} p_{jj}(2n) < \infty \quad \Longrightarrow \quad \text{State } j \text{ transient}$$