Summary We have seen

(i) If $\sum_{n=1}^{\infty} p_{ii}(n) = \infty$

$\Rightarrow$ state i is persistent

(ii) If $\sum_{n=1}^{\infty} p_{ii}(n) < \infty$

$\Rightarrow$ state i is transient

(iii) For SRW,

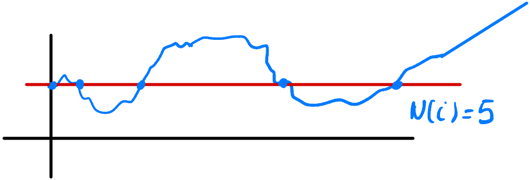
$p = \frac{1}{2} \Rightarrow p_{ii}(n) \sim \frac{c}{n^{\frac{1}{2}}} \Rightarrow \sum_n p_{ii}(n) = \infty$

$\Rightarrow \forall i \in \mathbb{Z}$, i is persistent

$p = \frac{1}{2} \Rightarrow p_{ii}(n) \sim c \frac{(c_p)^n}{n^{\frac{1}{2}}} \Rightarrow \sum p_{ii}(n) < \infty$

$\Rightarrow$ i transient

Number of visits



Recall: We have seen that

State j is either persistent or transient

Number of visits: We set

$$N(i) = \# \text{ times that } X \text{ visits its starting point } i$$

Fact: We have

$$\mathbf{P}\left(N(i)=\infty \mid X_0=i\right) = \begin{cases} 1\,, & \text{if } i \text{ persistent} \\ 0\,, & \text{if } i \text{ transient} \end{cases}$$

Behavior of T_j for a transient state

Recall: We set $T_j = \infty$ if there is no visit to j, and

$$T_j = \inf\{n \geq 1;\, X_n = j\}$$

Mean for T_j if j is transient: Whenever j is transient,

definition of transient

$$\begin{aligned} \mathbf{P}\left(T_j = \infty \mid X_0 = j\right) &> 0 \\ \mathbf{E}\left[T_j \mid X_0 = j\right] &= \infty \end{aligned}$$

Fact If Y is a random variable with values in $\mathbb{N} \cup \{\infty\}$

Then
$$\boxed{\begin{array}{l} \overbrace{P(Y=\infty)}^{=\gamma} > 0 \\ \Longrightarrow \quad E[Y] = \infty \end{array}}$$

"Proof"
$$E[Y] = \overbrace{\sum_{i=1}^{\infty} i\, P(Y=i)}^{\geq 0} + \text{"}\gamma \times \infty\text{"}$$
$$\geq \text{"}\gamma \times \infty\text{"} = \infty$$

Application If j transient

$$\Longrightarrow P(T_j = \infty \mid X_0 = j) > 0$$
$$\Longrightarrow E[T_j \mid X_0 = j] = \infty$$

Mean recurrence time

Definition 17.

Let

- X Markov chain
- i state in S

$$= \sum_{n=1}^{\infty} n\, P(T_i = n \mid X_0 = i)$$

Then we set

$$\mu_i = \mathbf{E}\left[T_i \middle| X_0 = i\right] = \begin{cases} \sum_{n=1}^{\infty} n\, f_{ii}(n)\,, & \text{if } i \text{ is persistent} \\ \infty\,, & \text{if } i \text{ is transient} \end{cases}$$

Null and positive states

Definition 18.

Let

- X Markov chain
- i persistent state in S, with mean recurrence time μ_i

Then

1. i is said to be null if $\mu_i = \infty$
2. i is said to be positive if $\mu_i < \infty$

Criterion for nullity

Theorem 19.

Let

- X Markov chain
- i persistent state in S

Then

$$i \text{ is null iff } \lim_{n\to\infty} p_{ii}(n) = 0$$

Rmk

State i is persistent null iff

$$\sum_{n=1}^{\infty} p_{ii}(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} p_{ii}(n) = 0$$

Typical example of u_n s.t

$$\lim u_n = 0 \quad , \quad \sum_n u_n = \infty$$

$$u_n = \frac{1}{n^\alpha} \qquad \alpha \le 1$$

Period

Definition 20.

Let

- X Markov chain, i state in S

Then

1. The period of i is given by
$$d(i) = \gcd\left\{n;\ p_{ii}(n) > 0\right\}$$
2. The state i is aperiodic if $d(i) = 1$, periodic if $d(i) > 1$

Interpretation: The period describes
$\hookrightarrow$ Times at which returns to i are possible

Ergodic states

Definition 21.

Let

- X Markov chain
- i state in S

Then i is said to be ergodic if

$$i \text{ is persistent, positive and aperiodic}$$

Ergodic :

- $P(N(i) = \infty \mid X_0 = i) = 1$
- $E[T_i \mid X_0 = i] < \infty$
- $p_{ii} = P(X_1 = i \mid X_0 = i) > 0$

Simple random walk case

Proposition 22.

Let

- X simple random walk
- Parameters p and $q = 1 - p$

Then the states are

1. Periodic with period 2
2. Transient if $p \neq \frac{1}{2}$
3. Null persistent if $p = \frac{1}{2}$

Let X SRW

(i) Period $= 2$. We have seen that

$p_{ii}(2n) > 0$

$p_{ii}(2n+1) = 0$

Thus $\gcd\{k ; p_{ii}(k) > 0\}$

$= \gcd\{\text{even numbers}\}$

$= 2$

(ii) X transient if $p \neq \frac{1}{2}$

$\rightarrow$ seen on W

(iii) X persistent if $p = \frac{1}{2}$ $\longrightarrow$ seen on w

(iv) X null persistent if $p = \frac{1}{2}$

we have seen that i null persistent if

$$\sum_{n \geq 1} p_{ii}(n) = \infty \quad , \quad \lim_{n \to \infty} p_{ii}(n) = 0$$

Here for $p = \frac{1}{2}$ we have

$$p_{ii}(n) \sim \frac{c}{n^{\frac{1}{2}}} \Rightarrow \sum_n p_{ii}(n) = \infty$$

$$\lim_{n \to \infty} p_{ii}(n) = 0$$

$\Rightarrow$ $\boxed{i \text{ null persistent } \forall i \in \mathbb{Z}}$

Proof of Proposition 22 (1)

Transience if $p \neq \frac{1}{2}$:
This has been established in Proposition 16

Null recurrence if $p = \frac{1}{2}$:

- This has been established
  $\hookrightarrow$ in Generating functions - Proposition 12
- We have seen that $\mathbf{E}[T_0] = \infty$

Proof of Proposition 22 (2)

Another way to look at null recurrence: If $p = \frac{1}{2}$ we have seen

$$p_{ii}(2n) \sim \frac{1}{(\pi n)^{1/2}}, \qquad p_{ii}(2n+1) = 0$$

Hence

$$\lim_{n\to\infty} p_{ii}(n) = 0$$

According to Theorem 19, i is recurrent null

Period 2: The fact that $d(i) = 2$ stems from

$$p_{ii}(2n) > 0, \qquad p_{ii}(2n+1) = 0$$

Branching process case

$X_{n+1} = \varphi(X_n, Z_{n+1})$

Proposition 23.

Consider a branching process with

- $Z_1 \sim f$, f with pgf G
- $\mathbf{P}(Z_1 = 0) = f(0) > 0$

Then

1. 0 is an absorbing state:
$$\mathbf{P}(X_n = 0 \text{ for all } n \mid X_0 = 0) = 1$$
2. Other states are transient

Conclusion: There are only 2 possibilities
(i) Extinction
(ii) Explosion