Outline

Communication

Recall: For a Markov chain X, we have seen that

$$\mathbf{P}\left(X_n = j \mid X_0 = i\right) = p_{ij}(n) = P^n(i,j)$$

Communication:
We say that i communicates with j if

$$\text{There exists } n \geq 0 \text{ such that } \underbrace{\mathbf{P}\left(X_n = j \mid X_0 = i\right) = p_{ij}}_{\text{My chances to reach } j \text{ from } i \text{ in } n \text{ steps are } > 0}(n) > 0.$$

Notation: $i \to j$.

Intercommunication

Intercommunication:

If $i \to j$ and $j \to i$, we say that i and j intercommunicate.

Notation: $i \leftrightarrow j$.

Remarks:

Due to the fact that $P^0 = Id$

1. For all $i \in S$, we have $i \leftrightarrow i$, since $p^0(i,i) = 1$.
2. If $i \to j$ and $j \to k$, then $i \to k$.

Graph related to a Markov chain

Definition 24.

Let X be a Markov chain with transition p.
We define a graph $\mathcal{G}(X)$ given by

- $\mathcal{G}(X)$ is an oriented graph
- The vertices of $\mathcal{G}(X)$ are points in S.
- The edges of $\mathcal{G}(X)$ are given by the set

$$\mathbb{V} \equiv \{(i,j);\ i \neq j,\ p(i,j) > 0\}.$$

Example

Definition of the chain: Take $S = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

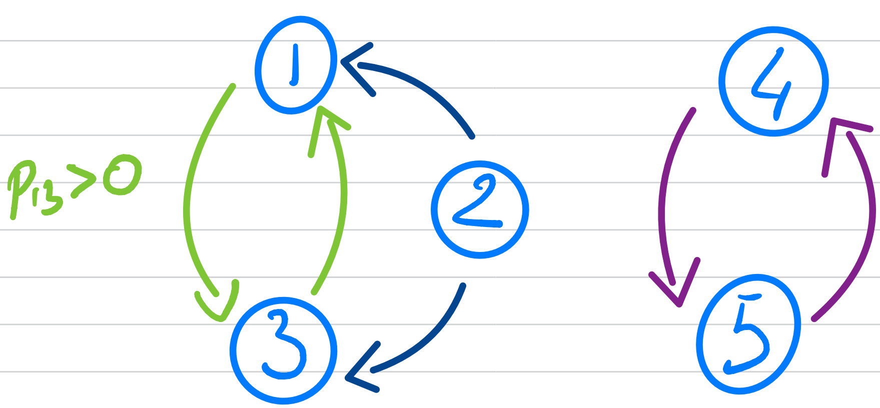
Related graph: to be done in class

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

(ii) If we start from 2, we will at some point jump to 1 or 3 and stay there forever

etc...

$g(x) =$



<u>Intuitively</u> : (i) If we start from 1, we will loop between 1 & 3

Recall If $i \neq j$, $i \to j$ iff $\exists\, n$ s.t.
$P^n(i,j) = p_{ij}(n) > 0$

Writing the n-th power

$$P^n(i,j) = \sum_{i_1,\dots,i_{n-1} \in S} p(i,i_1)\, p(i_1,i_2) \cdots p(i_{n-1},j)$$

If $P^n(i,j) > 0$, $\exists\, (i_1, \dots, i_{n-1})$ s.t.
$p(i,i_1) \cdots p(i_{n-1},j) > 0$

Thus $p(i,i_1) > 0$, $p(i_1,i_2) > 0, \dots, p(i_{n-1},j) > 0$

$\Rightarrow$ $\boxed{\text{Path } (i,i_1), (i_1,i_2), \dots, (i_{n-1},j) \text{ in the set } V \text{ of } g(x)}$

Proof of Proposition 25

Relation with the graph: If $i \neq j$ we have

$(i \to j) \Leftrightarrow$ There exists $n \geq 1$ such that $p_{ij}(n) > 0$

$\Leftrightarrow$ There exists $n \geq 1$ such that

$$\sum_{i_1,\ldots,i_{n-1} \in E} p_{i,i_1} \cdots p_{i_{n-1},j} > 0$$

$\Leftrightarrow$ There exists $n \geq 1$ et $i_1, \ldots, i_{n-1} \in E$ such that

$$p_{i,i_1} \cdots p_{i_{n-1},j} > 0$$

$\Leftrightarrow$ There exists an oriented path from i to j in $\mathcal{G}(X)$

Irreducible classes

If C_k is irreducible, all states in C_k intercommunicate

Proposition 26.

Let

- X Markov chain with transition p

Then

1. The relation $\leftrightarrow$ is an equivalence relation.
2. Denote $C_1, \ldots, C_l$ the equivalence classes for $\leftrightarrow$ in S. Then $\rightarrow$ is a partial order relation between classes:
$$C_1 \rightarrow C_2 \text{ and } C_2 \rightarrow C_3 \Longrightarrow C_1 \rightarrow C_3$$
3. $C_1 \rightarrow C_2$ iff $\exists\ i \in C_1$ and $j \in C_2$ such that $i \rightarrow j$.
4. The classes are called irreducible

Equivalence relation

(i) **Reflexive** $\quad i \leftrightarrow i$

(ii) **Symmetric** $\quad$ if $i \leftrightarrow j$

then $j \leftrightarrow i$

(iii) **Transitive** $\quad$ if $i \leftrightarrow j$

and $j \leftrightarrow k$

then $i \leftrightarrow k$

Def $\quad i, j$ are in the same class fn $\leftrightarrow$

if $i \leftrightarrow j$

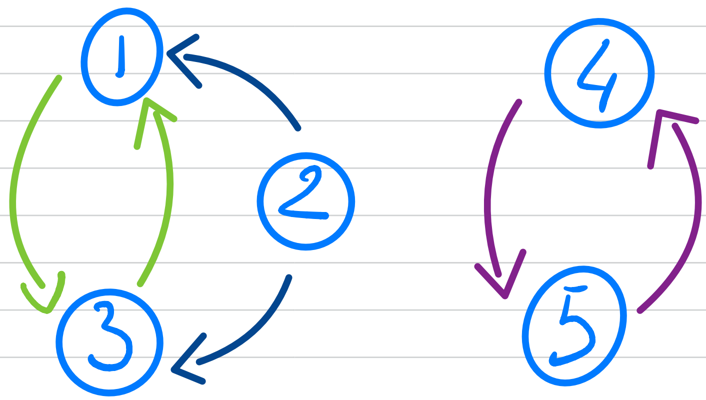
Example (1)

Definition of the chain: Take $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

$P^0 = Id \Rightarrow i \leftrightarrow i$

Graph



classes

$C_1 = \{1, 3\}$ $C_2 = \{2\}$ $C_3 = \{4, 5\}$

Communication

$C_2 \longrightarrow C_1$, since $2 \in C_2 \longrightarrow 3 \in C_1$

On W: How to use that in order to see if states are transient / persistent