Nature of intercommunicating states

Theorem 27.

Let

- X Markov chain with transition p
- i, j such that $i \leftrightarrow j$

Then

1. i, j have the same period
2. i transient iff j transient
3. i null persistent iff j null persistent

Claim If $i \leftrightarrow j$ and i transient, then j is transient

Recall i transient iff $\sum_{r=0}^{\infty} P_{ii}(r) < \infty$

Here $i \to j$: $\exists\, m$ s.t. $P_{ij}(m) > 0$

$j \to i$: $\exists\, n$ s.t. $P_{ji}(n) > 0$

Then for $r \geq 1$

$$P_{ii}(m+n+r) = \sum_{i_1, i_2 \in S} P_{i i_1}(m)\, P_{i_1 i_2}(r)\, P_{i_2 i}(n)$$

$$\overset{i_1 = i_2 = j}{\geq} P_{ij}(m)\, P_{jj}(r)\, P_{ji}(n)$$

$$= \overbrace{P_{ij}(m)\, P_{ji}(n)}^{\equiv \alpha > 0}\, P_{jj}(r)$$

i transient

Summary : We have obtained

$$\infty > \sum_{r=1}^{\infty} P_{ii}(m+n+r) \geq \alpha \sum_{r=1}^{\infty} P_{jj}(r)$$

Thus

$$\sum_{r=1}^{\infty} P_{jj}(r) < \infty$$

$$\Rightarrow \boxed{j \text{ transient}}$$

Proof of Theorem 27 – item 2 (1)

A positive quantity: If $i \leftrightarrow j$, then there exists $m, n \geq 1$ such that

$$\alpha \equiv p_{ij}(m) p_{ji}(n) > 0$$

Application of Chapman-Kolmogorov: We get

$$p_{ii}(m + r + n) \geq p_{ij}(m) p_{jj}(r) p_{ji}(n) = \alpha \, p_{jj}(r)$$

Summing over r: We get

$$\sum_{r=0}^{\infty} p_{ii}(r) < \infty \quad \Longrightarrow \quad \sum_{r=0}^{\infty} p_{jj}(r) < \infty$$

Proof of Theorem 27 – item 2 (2)

Conclusion:

$$i \text{ transient} \quad \Longrightarrow \quad j \text{ transient}$$

Closed class

Interpretation: If you are in C, you cannot get out of C

Definition 28.

An equivalent class C is closed if:

For all $i \in C$ and $j \notin C$, we have $i \not\to j$.

Some rules for closedness:

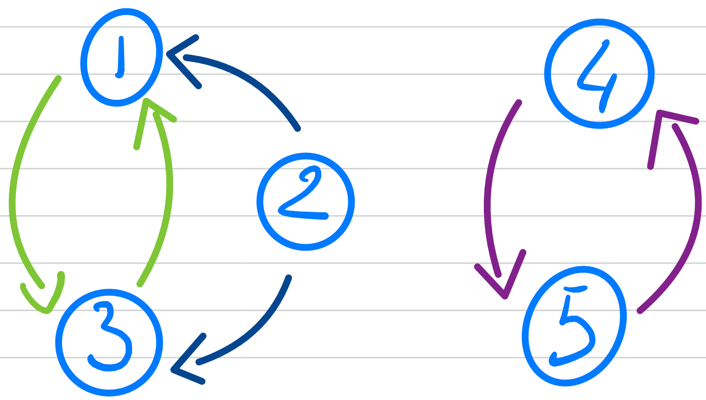
- If there exists a unique class C, it is closed
- There exists a unique closed class C
  $\Leftrightarrow$ There exists a class C s.t for all $i \in \cancel{E}\ S$, we have $i \to C$.

Example ctd (1)

Definition of the chain: Take $E = \{1, 2, 3, 4, 5\}$ and

$$p = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2/3 & 1/3 \end{pmatrix}$$

Graph



classes

$C_1 = \{1, 3\}$ $C_2 = \{2\}$ $C_3 = \{4, 5\}$

closed classes

C_3 , C_1

Example ctd (2)

Recall: The related classes are
$C_1 = \{1, 3\}$, $C_2 = \{2\}$ and $C_3 = \{4, 5\}$.
We have $C_2 \to C_1$

Closed classes: We find

$$C_1, C_3 \text{ closed, and } C_2 \text{ not closed}$$

Random walk example

Proposition 29.

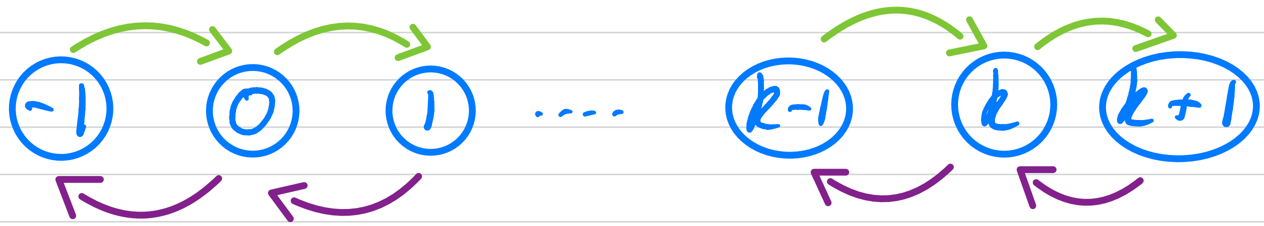
Let

- X simple random walk
- Parameters p and $q = 1 - p$

Then

1. There is a unique class, $C = \mathbb{Z}$
2. This class is closed
3. If one state is transient, all the states are transient
4. If one state is null pers., all the states are null pers.
5. All the states have the same period $(=2)$

Graph for the rw



This graph is connected.

Thus for all $i, j \in \mathbb{Z}$ we have

$$i \longleftrightarrow j$$

$$\Rightarrow \quad \boxed{\begin{array}{l}\text{There is a unique class} \\ \quad C_1 = \mathbb{Z}\end{array}}$$

Rmk 1 We have seen (ch. on pgf's) that

0 is transient if $p \neq \frac{1}{2}$

0 is null recurrent if $p = \frac{1}{2}$

Now we get that $\forall j \in \mathbb{Z}$

$$\boxed{\begin{array}{l} j \text{ is transient if } p \neq \tfrac{1}{2} \\ j \text{ is null recurrent if } p = \tfrac{1}{2} \end{array}} \quad \begin{array}{l} (1) \\ (2) \end{array}$$

Rmk 2 We already knew (1) and (2).
We have seen $\forall i \in \mathbb{Z}$

$$\sum_{n \geq 1} p_{ii}(n) < \infty \text{ if } p \neq \tfrac{1}{2} \qquad \left.\begin{array}{l} \sum p_{ii}(n) = \infty \\ p_{ii}(n) \longrightarrow 0 \end{array}\right\} \text{ if } p = \tfrac{1}{2}$$

Decomposition theorem

Theorem 30.

Let

- X Markov chain with transition p
- S state space

Then S can be partitioned uniquely as

$$S = T \cup C_1 \cup C_2 \cup \cdots,$$

where

- $T \equiv$ Set of transient states
- $C_k \equiv$ irreducible closed class of persistent states

Finite state space case

Proposition 31.

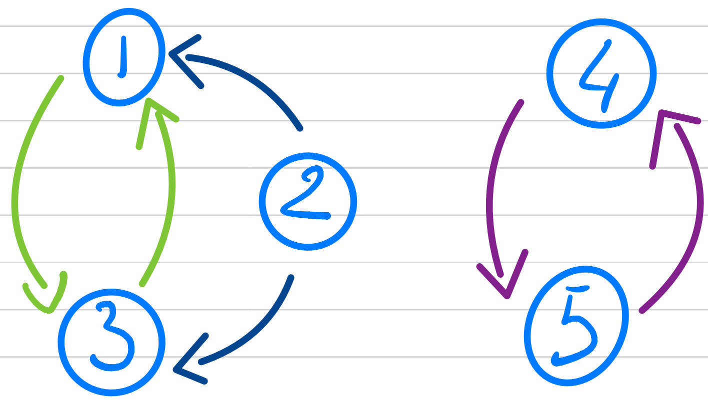
Let

- X Markov chain with transition p
- S finite state space with $S = T \cup C_1 \cup C_2 \cup \cdots$

Then

1. At least 1 state in S is persistent
2. All persistent states are positive $E[T_i \mid X_0 = i] < \infty$
3. Later we will see: every state in C_k is positive persistent

Graph



classes

$C_1 = \{1,3\}$ $C_2 = \{2\}$ $C_3 = \{4,5\}$

closed classes

C_3 , C_1

Here $S = \{1,..,5\}$ finite

$\Rightarrow$ 2 is transient , 1, 3, 4, 5 are positive persistent