Convolution

Definition 2.

Let

- $a = \{a_i;\ i \geq 0\}$ and $b = \{b_i;\ i \geq 0\}$ sequences
- c sequence defined by

a_i

$$c_n = \sum_{i=0}^{n} c_i b_{n-i}$$

Then we denote

$$c = a * b$$

Convolution and generating functions

Proposition 3.

Let

- $a = \{a_i;\ i \geq 0\}$ and $b = \{b_i;\ i \geq 0\}$ sequences
- $c = a * b$

Then

$$G_c(s) = G_a(s)\, G_b(s)$$

Proof We consider

$$G_c(s) \stackrel{def}{=} \sum_{n=0}^{\infty} c_n s^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) s^n \quad = s^i s^{n-i}$$

$$= \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_i s^i b_{n-i} s^{n-i}$$

Here $0 \le i \le n < \infty$

This can also be written $\begin{cases} 0 \le i < \infty \\ i \le n < \infty \end{cases}$. Thus

$$\boxed{G_c(s)} = \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} a_i s^i b_{n-i} s^{n-i}$$

$$m := n-i$$

$$= \sum_{i=0}^{\infty} a_i s^i \sum_{n=i}^{\infty} b_{n-i} s^{n-i} = \sum_{i=0}^{\infty} a_i s^i \sum_{m=0}^{\infty} b_m s^m \quad \boxed{= G_a(s)\, G_b(s)}$$

Proof of Proposition 3

Computation from the definition of G_c: We have

$$\begin{aligned} G_c(s) &= \sum_{n=0}^{\infty} c_n s^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) s^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_i s^i \, b_{n-i} s^{n-i} \\ &= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} a_i s^i \, b_{n-i} s^{n-i} \\ &= G_a(s) \, G_b(s) \end{aligned}$$

Recall: Poisson distribution.

This distribution is defined on $\{0,1,2,\dots\}$ by

$(X \sim \mathcal{P}(\lambda))$

$$a_k = P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

Poisson random variable (1)

If $X \sim \mathcal{P}(\lambda)$

then $X \in \{0, 1, \dots\}$

state space

Notation:

$$\mathcal{P}(\lambda) \text{ for } \lambda \in \mathbb{R}_+$$

State space:

$X : \Omega \longrightarrow$ state space

$$E = \mathbb{N} \cup \{0\}$$

Pmf:

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0$$

Expected value, variance and pgf: probability generating function

$$\mathbf{E}[X] = \lambda, \qquad \mathbf{Var}(X) = \lambda, \qquad G_X(s) = \exp(\lambda(s-1))$$

Computation of $G_X(s)$ For $X \sim P(\lambda)$

$$G_X(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda s)^k}{k!}$$

$$= e^{-\lambda} e^{\lambda s}$$

$$= e^{-\lambda + \lambda s}$$

$$\boxed{G_X(s) = e^{\lambda(s-1)}}$$

Poisson random variable (2)

Use (examples):

- # customers getting into a shop from 2pm to 5pm
- # buses stopping at a bus stop in a period of 35mn
- # jobs reaching a server from 12am to 6am

Empirical rule:
If $n \to \infty$, $p \to 0$ and $np \to \lambda$, we approximate $\text{Bin}(n, p)$ by $\mathcal{P}(\lambda)$.
This is usually applied for

$$p \leq 0.1 \quad \text{and} \quad np \leq 5$$

Poisson random variable (3)

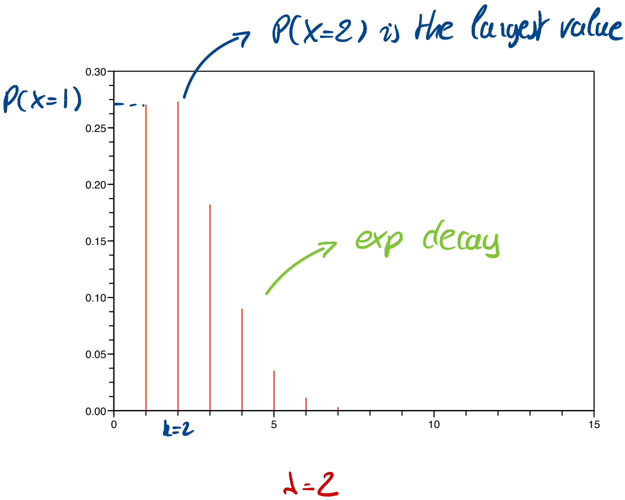
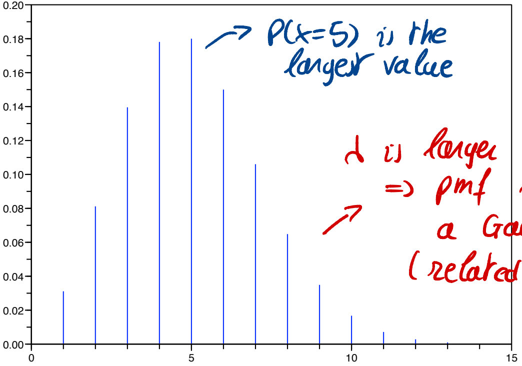


Figure: Pmf of $\mathcal{P}(2)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Poisson random variable (4)



$P(X=5)$ is the largest value

λ is larger
$\Rightarrow$ pmf looks more like a Gaussian curve
(related to CLT)

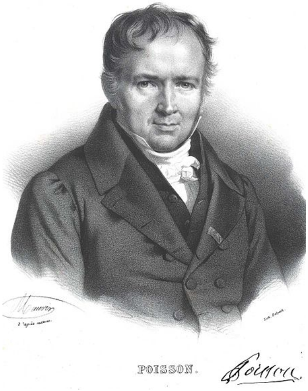
$\lambda = 5$

Figure: Pmf of $\mathcal{P}(5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Siméon Poisson

Some facts about Poisson:

- Lifespan: 1781-1840, in $\simeq$ Paris
- Engineer, Physicist and Mathematician
- Breakthroughs in electromagnetism
- Contributions in partial diff. eq celestial mechanics, Fourier series
- Marginal contributions in probability



A quote by Poisson:

Life is good for only two things: doing mathematics and teaching it!!

We know: If $X \perp\!\!\!\perp Y$ and $Z = X+Y$ then

$$f_Z = f_X * f_Y \quad , \text{where } f_Z = \text{pmf of } Z$$

Thus

$$\begin{aligned} G_Z(s) &= G_X(s)\, G_Y(s) \\ &= e^{\lambda(s-1)}\, e^{\mu(s-1)} \\ &= \underbrace{e^{(\lambda+\mu)(s-1)}}_{\text{pgf of } P(\lambda+\mu)} \end{aligned}$$

Thus $\boxed{Z \sim P(\lambda+\mu)}$