Outline

Stationary distribution

Definition 32.

Let

- X Markov chain with matrix transition P
- π vector (row vector) $\pi = (\pi_1, \pi_2, \ldots, \ )$

Then π is a stationary distribution if

1. $(\pi_j \geq 0$ for all $j \in S$ and $\sum_{j\in S} \pi_j = 1) \rightarrow$ π is a (probab.) distribution
2. π satisfies $\pi = \pi P$, that is

$$\pi_j = \sum_{i\in S} \pi_i p_{ij}, \quad \text{for all} \quad j \in S$$

Interpretation of stationary distribution

Proposition 33.

Let

stationary

- X Markov chain with matrix transition P
- π invariant distribution

The values of $n \mapsto X_n$ fluctuate but the distribution of X_n is constant

Then

$$X_0 \sim \pi \quad \Longrightarrow \quad \overbrace{X_n \sim \pi} \quad \text{for all } n \geq 0$$

Otherwise stated,

$$\mathbf{P}\left(X_n = j \middle| X_0 \sim \pi\right) = \pi_j$$

Definition $P(X_1 = j \mid X_0 \sim \pi)$ means

$$P(X_1 = j \mid X_0 \sim \pi) = \sum_{i \in S} P(X_1 = j \mid X_0 = i)\, \pi_i$$

Expression in terms of P

$$P(X_1 = j \mid X_0 \sim \pi) = \sum \pi_i \, P_{ij}$$

$$= (\pi P)_j$$

If π is stationary, $(\pi P)_j = \pi_j$

$$\Rightarrow \quad P(X_1 = j \mid X_0 \sim \pi) = (\pi P)_j = \pi_j$$

$\Rightarrow$ π is the distribution of X_1

From n to $n+1$

Hyp: $P(X_n = j \mid X_0 \sim \pi) = \pi_j \quad \forall j \in S$

i.e $\pi P^n = \pi$

$$P(X_{n+1} = j \mid X_0 \sim \pi)$$

$$= (\pi P^{n+1})_j$$

$$= (\underbrace{\pi P^n}_{=\pi} P)_j$$

$$= (\pi P)_j$$

$$= \pi_j$$

Thus $\boxed{X_n \sim \pi \quad \text{for all } n}$ by induction

Proof of Proposition 33

Distribution of X_1: We have

$$\begin{aligned}
\mathbf{P}\left(X_1 = j \middle| X_0 \sim \pi\right) &= \sum_{i \in S} \mathbf{P}\left(X_1 = j \middle| X_0 = i\right) \pi_i \\
&= (\pi P)_j \\
&= \pi_j
\end{aligned}$$

Distribution of X_n: Use a recursion and

$$\mathbf{P}\left(X_{n+1} = j\right) = \sum_{i \in S} \mathbf{P}\left(X_{n+1} = j \middle| X_n = i\right) \mathbf{P}\left(X_n = i\right)$$

Stationary distributions and persistent chains

Rmk: $\pi P = \pi$ is a linear system of equations

Theorem 34.

Let

- X Markov chain with matrix transition P
- X irreducible

Then

X has a stationary distribution

$\Longleftrightarrow$

All states are non-null persistent

Stationary distributions and return times

Theorem 35.

Let

- X Markov chain with matrix transition P
- X irreducible
- X admits a stationary distribution π

Then

$$\pi_i = \frac{1}{\mu_i} = \frac{1}{\mathbf{E}[T_i | X_0 = i]}$$

Intuition If we have a stationary distribution π and $X_0 \sim \pi$, it reasonable to think that

proportion of time (for long periods) spent at state i $\simeq \pi_i$

$\Rightarrow$ the time we need to go back to i is $\simeq \frac{1}{\pi_i}$

We get $\boxed{E[T_i \mid X_0 = i] = \frac{1}{\pi_i}}$

Rmk Thm 35 states something like

$E\left[\frac{1}{Y}\right] \simeq \frac{1}{E[Y]}$ $\longrightarrow$ Wrong in general!

Hints about the proof

Main ingredient: Prove that

$$\mu_k = \sum_{i \in S} \rho_i(k), \quad \text{with} \quad \rho_i(k) = \sum_{n=1}^{\infty} \mathbf{P}\left(X_n = i, T_k \geq n \middle| X_0 = k\right),$$

is solution to $\mu = \mu P$

Idea for $\pi_i = (\mu_i)^{-1}$: One writes

$$\begin{aligned} \pi_i &= \text{"Average time spent at } i\text{"} \\ &\simeq \frac{1}{\text{"Average time to return at } i\text{"}} \end{aligned}$$

Example (1)

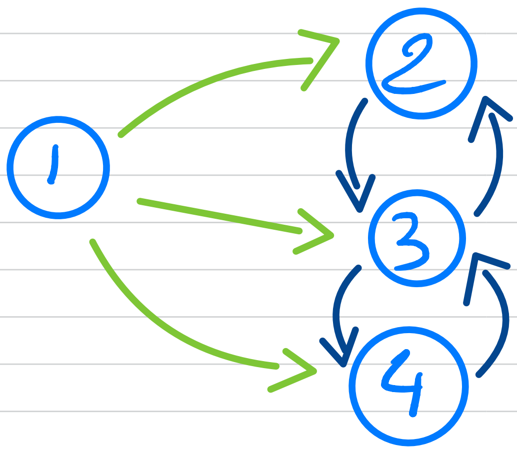
Definition of the chain: Take $S = \{1,2,3,4\}$ (hence $|S| < \infty$) and

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Chain on $S = \{1,2,3,4\}$

Monday: what else can we say with π?

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



classes

$C_1 = \{1\}$ not closed

$C_2 = \{2,3,4\}$ closed

Classification of states

1 is transient $\qquad$ 2, 3, 4 are persistent