Criterion for positivity/nullity

Theorem 36.

Let

- X Markov chain with matrix transition P
- X irreducible
- X recurrent

Then

1. There exists a measure x satisfying $x = x\,P$
2. x is unique up to multiplicative constant
3. x has strictly positive entries
4. The chain is positive if $\sum_{i\in S} x_i < \infty$
5. The chain is null if $\sum_{i\in S} x_i = \infty$

Criterion for transience

Theorem 37.

Let

- X Markov chain with matrix transition P
- X irreducible
- s any state in S

Then

$$X \text{ is transient}$$
$$\Longleftrightarrow$$
$$\text{There exists a non zero solution } \{y_i;\, i \neq s\}$$
$$\text{to } y_i = \sum_{j \neq s} p_{ij} y_j, \text{ with } |y_i| \leq 1$$

Random walk with retaining barrier (1)

Model: Random walk on $\mathbb{N}$
$\hookrightarrow$ With retaining barrier at 0

Transition probability: We get

$$p_{00} = q, \quad p_{i,i+1} = p, \text{ if } i \geq 0, \quad p_{i,i-1} = q, \text{ if } i \geq 1$$

Notation: We set

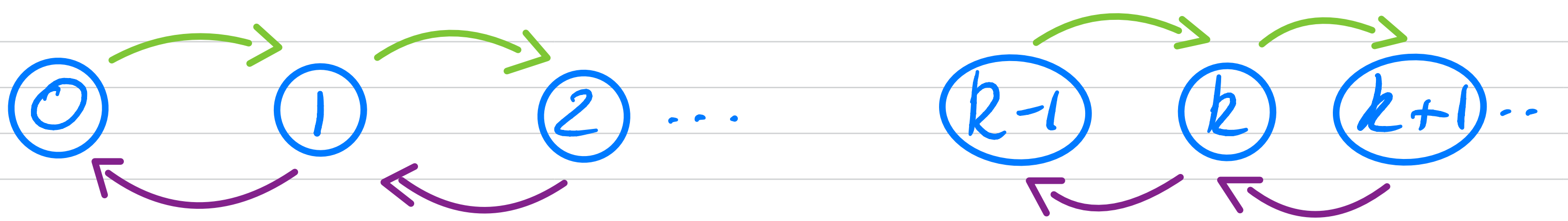
$$\rho = \frac{p}{q}$$

$P(X_0=i \mid X_0=i)=1 \Rightarrow p^0(i,i)>0 \Rightarrow i \leftrightarrow i$

Graph for X

$P_{00} = q \qquad P_{01} = p$

For $i \geq 1$ $\qquad P_{i,i-1} = q \qquad P_{i,i+1} = p$



we have $i \leftrightarrow j$ for all $i,j \in \mathbb{N}$

$\Rightarrow \boxed{X \text{ irreducible}}$

Random walk with retaining barrier (2)

$\rho = \frac{p}{q}$

Proposition 38.

Let X be the random walk with retaining barrier. Then

1. If $p > \frac{1}{2}$, the chain is transient
2. If $p < \frac{1}{2}$, the chain is non-null persistent
   $\hookrightarrow$ with stationary distribution given by

$$\pi = \text{Nbin}(1, 1 - \rho)$$

3. If $p = \frac{1}{2}$, the chain is null persistent

Intuition

(i) If $p > \frac{1}{2}$, the SRW Z_n is such that

$$Z_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty$$

Thus for large enough n, the "bouncing" at 0 does not play a role

$\Rightarrow$ X_n is also s.t. $\boxed{\lim_{n\to\infty} X_n = \infty}$

(ii) If $p < \frac{1}{2}$, the SRW Z_n wants to go to $-\infty$. The barrier at 0 prevents this

$\Rightarrow$ we hit 0 an ∞ number of times $\Rightarrow$ $\boxed{X_n \text{ is persistent}}$

Case $p > \frac{1}{2}$. We have that

$$y_i = 1 - \frac{1}{\rho^i} \quad \text{solve} \quad y_i = \sum_{j \neq i} p_{ij} \, y_j$$

Moreover $\rho = \frac{p}{q} \overset{p > \frac{1}{2}}{>} 1 \implies |y_i| \leq 1$

Thus we get that $\boxed{X \text{ transient}}$

Claim

$y_i = 1 - \frac{1}{\rho^i}$ solve $y_i = \sum_{j \neq i} p_{ij} \, y_j$

For $i \geq 1$,

$$\boxed{\sum_{j \neq i} p_{ij} \, y_j} = p_{i,i-1} \, y_{i-1} + p_{i,i+1} \, y_{i+1}$$

$$= q \left(1 - \frac{1}{\rho^{i-1}}\right) + p \left(1 - \frac{1}{\rho^{i+1}}\right)$$

$$= \overbrace{q + p}^{=1} - \left\{ q \times \frac{1}{\rho^{i-1}} + p \times \frac{1}{\rho^{i+1}} \right\}$$

$$= 1 - \frac{1}{\rho^{i+1}} \left\{ q \rho^2 + p \right\}$$

$$= 1 - \frac{1}{\rho^{i+1}} \left\{ q \frac{p^2}{q^2} + p \right\}$$

$$= 1 - \frac{p}{\rho^{i+1}} \left\{ \overbrace{\frac{p}{q} + 1}^{= \frac{1}{q}} \right\}$$

$$= 1 - \frac{p}{\rho^{i+1}} \times \frac{1}{q} = 1 - \frac{1}{\rho^i} \quad \boxed{= y_i}$$

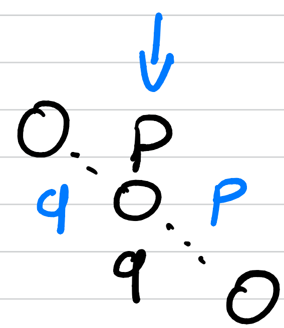
<u>Case $p < \frac{1}{2}$</u> $\quad \pi = \mathrm{Nbin}(1, 1-\rho)$ is invariant

$\Rightarrow \quad X$ non-null persistent

<u>Recall</u> that $\pi_k = \rho^k (1-\rho) \quad \forall k \geq 0$

<u>Claim</u> : $\pi P = \pi$ $\quad$ when $\quad q > p$

<u>If $j \geq 1$</u> ,



$$(\pi P)_j = \sum_i \pi_i P_{ij} =$$

$$= \pi_{j-1}\, p + \pi_{j+1}\, q$$

$$= \rho^{j-1}(1-\rho)\, p + \rho^{j+1}(1-\rho)\, q$$

$$= \rho^{j-1}(1-\rho)\left\{ \overbrace{p + \rho^2 q}^{=\rho} \right\}$$

$$= \rho^{j}(1-\rho) = \pi_j$$

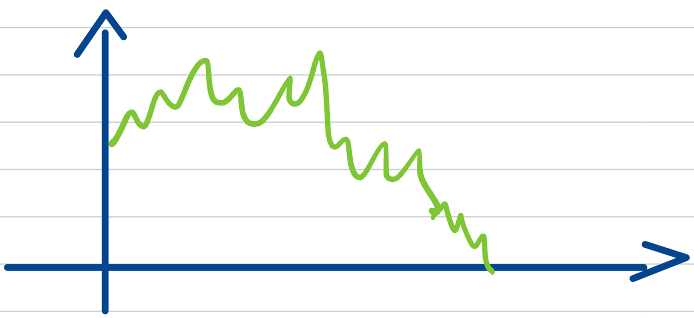
Case $p=\frac{1}{2}$

(i) X is persistent. Indeed, X stays longer at 0 than $Y_n = |Z_n|$ where $Z =$ SRW.

Indeed if P^X, P^Y, P^Z are the corresponding transitions, then for $i \geq 1$

$$P^X_{i,i-1} = P^Y_{i,i-1} = P^Z_{i,i-1} = q$$

$$P^X_{i,i+1} = P^Y_{i,i+1} = P^Z_{i,i+1} = p$$



and

$$P^Y_{00} = 0, \quad P^X_{00} = q$$

$$P^Y_{01} = 1, \quad P^X_{01} = p$$

Proof of Proposition 38 (1)

Case $q < p$: One verifies that

$$y_i = 1 - \rho^{-i} \quad \text{solves} \quad y_i = \sum_{j \neq s} p_{ij} y_j$$

Thus X transient

Case $q > p$: One sees that

$$\pi = \text{Nbin}(1, 1 - \rho) \quad \text{is such that} \quad \pi P = \pi$$

Thus X non-null persistent

Proof of Proposition 38 (2)

Computation for $q < p$: For $i \geq 1$ we have

$$
\begin{aligned}
\sum_{j \neq i} p_{ij} y_j &= p_{i,i-1}\, y_{i-1} + p_{i,i+1}\, y_{i+1} \\
&= q\left(1 - \frac{1}{\rho^{i-1}}\right) + p\left(1 - \frac{1}{\rho^{i+1}}\right) \\
&= 1 - \frac{1}{\rho^{i+1}}\left(q\rho^2 + p\right) \\
&= 1 - \frac{1}{\rho^{i+1}}\left(\frac{p^2}{q} + p\right) \\
&= 1 - \frac{p}{\rho^{i+1}}\left(\frac{p}{q} + 1\right) \\
&= 1 - \frac{1}{\rho^{i}} \\
&= y_i
\end{aligned}
$$

Proof of Proposition 38 (3)

Nbin$(1, 1-\rho)$ distribution: Defined for $k \geq 0$ by

$$\pi_k = \rho^k (1-\rho)$$

Verifying $\pi P = \pi$ for $q > p$: For $j \geq 1$ we have

$$\begin{aligned}
\sum_{i \geq 0} \pi_i p_{ij} &= \pi_{j-1}\, p + \pi_{j+1}\, q \\
&= \rho^{j-1}(1-\rho)p + \rho^{j+1}(1-\rho)q \\
&= \rho^{j-1}(1-\rho)\left(p + \rho^2 q\right) \\
&= \rho^j (1-\rho) \\
&= \pi_j
\end{aligned}$$

Proof of Proposition 38 (4)

Case $q = p$: We have

1. X persistent since
   - $Y \equiv$ random walk is persistent
   - $X = |Y|$

2. X null-persistent since since $x = \mathbf{1}$ is such that

$$x = xP, \quad \text{and} \quad \sum_{i \in S} x_i = \infty$$