Outline

Main objective

long term prediction
starting from state i

Aim in this section:

1. Get expressions for

$$\lim_{n\to\infty} p_{ij}(n)$$

2. Link with stationary distributions

Problem with parity (1)

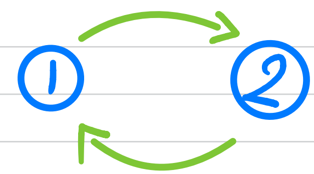
Example: Take $S = \{1, 2\}$ and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Question: Can we get

$$\lim_{n \to \infty} p_{ij}(n)\,?$$

Transition $\qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



x irreducible

Powers of P

$$P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad P^3 = P^2 \cdot P = P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By induction

$$P^{2n} = Id_{\mathbb{R}^2} \qquad P^{2n+1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In particular

$$p_{22}(2n) = 1 \qquad p_{22}(2n+1) = 0$$

$\hookrightarrow$ $\boxed{\text{no limit!}}$ $\qquad \hookrightarrow$ periodicity is 2

Problem with parity (2)

Example: Take $S = \{1, 2\}$ and

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Behavior of P^n and parity: We find

$$p_{11}(n) = p_{22}(n) = \begin{cases} 0\,, & \text{if } n \text{ is odd} \\ 1\,, & \text{if } n \text{ is even} \end{cases}$$

Thus

$p_{ii}(n)$ does not converge

Problem comes from periodicity

Aperiodic assumption

Hypothesis 39.

Until further notice we assume

$$\underbrace{X \text{ is an irreducible and aperiodic Markov chain}}$$

period $d = 1$

(probab to stay put $p_{ii} > 0$)

Stationary distributions and return times

Theorem 40.

Let

- X Markov chain with matrix transition P
- X irreducible and aperiodic

Then for all i, j we have

$$\lim_{n \to \infty} p_{ij}(n) = \frac{1}{\mu_j} = \frac{1}{\mathbf{E}[T_j | X_0 = j]}$$

Recall: We have just seen (X irreducible, $d=1$)

$$\lim_{n\to\infty} p_{ij}(n) = \frac{1}{E[T_j \mid X_0=j]} \qquad (1)$$

Rmk 1 If chain is null persistent we get

$$E[T_j \mid X_0=j] \overset{\text{(null)}}{=} \infty \Rightarrow \lim_{n\to\infty} p_{ij}(n) = 0$$

according to (1)

(we have already seen this property)

Rmk 2 The rhs of (1) does not depend on the initial condition i

$\Rightarrow$ we forget the initial condition as $n \to \infty$

Some remarks (1)

Persistent null case: If the Markov chain X is persistent null then

$$\lim_{n\to\infty} p_{ij}(n) = 0$$

We had seen this result in Proposition 15

Forgetting the past: If the Markov chain X is non-null persistent then

$$\lim_{n\to\infty} \mathbf{P}\left(X_n = j \middle| X_0 = i\right) = \lim_{n\to\infty} p_{ij}(n) = 0 = \pi_j = \frac{1}{\mu_j}$$

Thus the initial condition is forgotten

Some remarks (2)

$$\lim_{n\to\infty} P(X_n = j \mid X_0 \sim \nu) = \pi_j \quad (\text{regardless of } \nu)$$

Case with initial distribution: Assume

- X is non-null persistent
- $X_0 \sim \nu$

Then

$$\lim_{n\to\infty} \mathbf{P}\left(X_n = j \mid X_0 \sim \cancel{\mu}^{\,\nu}\right) = \lim_{n\to\infty} \overbrace{\sum_{i\in S} \nu_i p_{ij}(n)}^{[\nu P^n]_j} = \frac{1}{\mu_j}$$

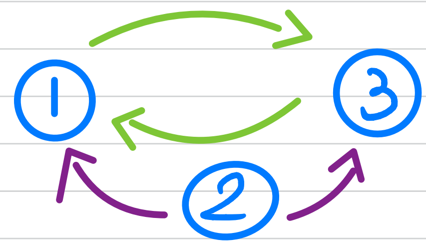
$p_{ij}(n) \to \frac{1}{\mu_j}$

dominated convergence

$$= \frac{1}{\mu_j} \overbrace{\sum_{i\in S} \nu_i}^{=1} = \frac{1}{\mu_j}$$

Example of transition on $S=\{1,2,3\}$

$$P=\begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$



Closed class: $\{1,3\} = C_1$, $T=\{2\}$

The invariant measure π is s.t. $\pi_2 = 0$

For $\hat{\pi} = [\pi_1, \pi_3]$ we solve

$$[\pi_1, \pi_3]\begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} = [\pi_1, \pi_3]$$

$$\iff \quad \frac{1}{3}\pi_1 + \frac{1}{2}\pi_3 = \pi_1 \quad \iff \quad \frac{2}{3}\pi_1 = \frac{1}{2}\pi_3$$

$$\iff \quad \pi_1 = \frac{3}{4}\pi_3$$

Computing $[\pi_1 \;\; \pi_3]$

$\pi_1 = \frac{3}{4}\pi_3$ & $\pi_1 + \pi_3 = 1$

we get

$$\frac{3}{4}\pi_3 + \pi_3 = 1 \iff \frac{7}{4}\pi_3 = 1 \iff \boxed{\begin{array}{l} \pi_3 = \frac{4}{7} \simeq .57 \\ \pi_1 = \frac{3}{7} \simeq .43 \end{array}}$$

Invariant measure :

$$\pi = [\,.43 \quad 0 \quad .57\,]$$

Rmk. We are not exactly under the condition of our theorem:

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

we have 2 classes

But $\boxed{d = 1}$ $(\, p_{ii} > 0,\ i = 1,2,3\,)$

Example

Definition of the chain: Take $S = \{1, 2, 3\}$ and

$$P = \begin{pmatrix} 1/3 & 0 & 2/3 \\ 1/4 & 1/2 & 1/4 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

Invariant measure: One finds

$$\pi = \begin{pmatrix} .43 & 0 & .57 \end{pmatrix}$$

Large time behavior: One finds (e.g with R)

we observe

$\lim_{n\to\infty} p_{ij}(n) = \pi_j$

$$P^{30} = \begin{pmatrix} .43 & 0 & .57 \\ .43 & 9 \times 10^{-10} & .57 \\ .43 & 0 & .57 \end{pmatrix}$$