# Outline

# Reversed chain

**Theorem 42.**

Let

- $X$ irreducible non-null persistent chain
- Transition for $X$ is $P$, invariant measure is $\pi$
- Hypothesis: $X_n \sim \pi$ for all $n$ $(\Leftarrow X_0 \sim \pi)$
- Set $Y_n = X_{N-n}$ for $0 \leq n \leq N$ (X with time run backwards)

Then

1. $Y$ is a Markov chain
2. The transition for $Y$ is

$$\mathbf{P}\left(Y_{n+1} = j \middle| Y_n = i\right) = \frac{\pi_j}{\pi_i} p_{ji}$$

Recall $\quad Y_n = X_{N-n} \quad$ for $\quad n = 0, \dots, N$

Quantity to evaluate : $\qquad Y_{n+1} = X_{N-(n+1)} = X_{N-n-1}$

$$Q_n = \boxed{\mathbb{P}\left( Y_{n+1} = i_{n+1} \mid Y_n = i_n, \dots, Y_0 = i_0 \right)}$$

$$= \frac{\mathbb{P}\left( Y_{n+1} = i_{n+1}, Y_n = i_n, \dots, Y_0 = i_0 \right)}{\mathbb{P}\left( Y_n = i_n, \dots, Y_0 = i_0 \right)}$$

$$= \frac{\mathbb{P}\left( X_{N-n-1} = i_{n+1}, X_{N-n} = i_n, \dots, X_N = i_0 \right)}{\mathbb{P}\left( X_{N-n} = i_n, \dots, X_N = i_0 \right)}$$

Markov trans.

$$= \frac{\pi_{i_{n+1}} \quad p_{i_{n+1}, i_n} \cdots\cdots \; p_{i_1, i_0}}{\pi_{i_n} \quad p_{i_n, i_{n-1}} \cdots\cdots \; p_{i_1, i_0}}$$

$$\boxed{= \frac{\pi_{i_{n+1}}}{\pi_{i_n}} \quad p_{i_{n+1}, i_n}}$$

Summary we have found

$$\mathbb{P}(Y_{n+1} = i_{n+1} \mid Y_n = i_n, \dots, Y_0 = i_0)$$

$$= \frac{\pi_{i_{n+1}}}{\pi_{i_n}} P_{i_{n+1}, i_n}$$

Now, with the same type of computation

$$\mathbb{P}(Y_{n+1} = i_{n+1} \mid Y_n = i_n) = \frac{\pi_{i_{n+1}}}{\pi_{i_n}} P_{i_{n+1}, i_n}$$

Thus $Y$ is a Markov chain with transition

$$\frac{\pi_j}{\pi_i} P_{ji}$$

# Proof of Theorem 42

Computing conditional probabilities: We have

$$
\begin{aligned}
&\mathbf{P}\left(Y_{n+1}=i_{n+1} \middle| Y_n=i_n,\ldots,Y_0=i_0\right) \\
&= \frac{\mathbf{P}\left(Y_0=i_0,\; Y_1=i_1,\ldots,Y_{n+1}=i_{n+1}\right)}{\mathbf{P}\left(Y_0=i_0,\; Y_1=i_1,\ldots,Y_n=i_n\right)} \\
&= \frac{\mathbf{P}\left(X_{N-n-1}=i_{n+1},\; X_{N-n}=i_n,\ldots,X_N=i_0\right)}{\mathbf{P}\left(X_{N-n}=i_n,\ldots,X_N=i_0\right)} \\
&= \frac{\pi_{i_{n+1}}\, p_{i_{n+1}i_n}\, p_{i_n i_{n-1}} \cdots p_{i_1 i_0}}{\pi_{i_n}\, p_{i_n i_{n-1}} \cdots p_{i_1 i_0}} \\
&= \frac{\pi_{i_{n+1}}\, p_{i_{n+1}i_n}}{\pi_{i_n}} \\
&= \mathbf{P}\left(Y_{n+1}=i_{n+1} \middle| Y_n=i_n\right)
\end{aligned}
$$

This gives the Markov property and the transition

# Reversed chain

**Definition 43.**

Let

- $X$ irreducible non-null persistent chain
- Transition for $X$ is $P$, invariant measure is $\pi$
- Hypothesis: $X_n \sim \pi$ for all $n$
- Set $Y_n = X_{N-n}$ for $0 \leq n \leq N$

Then transition $p_{ij}$ transition $\frac{\pi_j}{\pi_i} p_{ji}$

1. $X$ is said to be reversible if $Y$ has transition $P$
2. This is equivalent to

$$p_{ij} = \frac{\pi_j}{\pi_i} p_{ji}$$

$$\Longleftrightarrow \pi_i \, p_{ij} = \pi_j \, p_{ji}, \quad \text{for all} \quad i, j \in S$$

# Vocabulary

Detailed balance: Let

- $P$ transition matrix
- $\lambda$ distribution

Then $P, \lambda$ are in detailed balance if

$$\lambda_i \, p_{ij} = \lambda_j \, p_{ji}, \quad \text{for all} \quad i, j \in S$$

Reversible in equilibrium: If $X$ is such that

- $P, \pi$ are in detailed balance,

then $X$ is said to be reversible in equilibrium

# Invariant measure and reversibility

**Theorem 44.**

Let

- $X$ irreducible Markov chain with transition $P$
- Hypothesis: There exists a distribution $\pi$ such that

much easier to verify than $\pi P = \pi$

$$\pi_i \, p_{ij} = \pi_j \, p_{ji}, \quad \text{for all} \quad i, j \in S \tag{2}$$

Then

1. $\pi$ is a stationary distribution $\Rightarrow$ $X$ non-null persistent
2. $X$ is reversible in equilibrium

Claim If $\pi_i p_{ij} = \pi_j p_{ji}$, then $\pi$ is an invariant measure

Proof $[\pi P]_j = \sum_{i \in S} \pi_i p_{ij}$

detailed balance

$$= \sum_{i \in S} \pi_j p_{ji}$$

$$= \pi_j \overbrace{\sum_{i \in S} p_{ji}}^{=1} \quad (P \text{ stochastic matrix})$$

$$= \pi_j$$

Conclusion $[\pi P]_j = \pi_j$

$\Rightarrow$ $\boxed{\pi \text{ invariant}}$

# Proof of Theorem 44

Computation of $\pi P$: We have

$$\begin{aligned}(\pi P)_j &= \sum_{i \in S} \pi_i p_{ij} \\ &= \sum_{i \in S} \pi_j p_{ji} \\ &= \pi_j \sum_{i \in S} p_{ji} \\ &= \pi_j\end{aligned}$$

Conclusion:

1. $\pi$ is invariant
2. $X$ is reversible in equilibrium from (2)

# Ehrenfest diffusion model (1)

Model: We consider

- Two boxes $A$ and $B$
- Total of $N$ gas molecules in $A \cup B$
- At time $n$, one molecule is picked from the $N$ molecules
- This molecule changes box

Process: We set

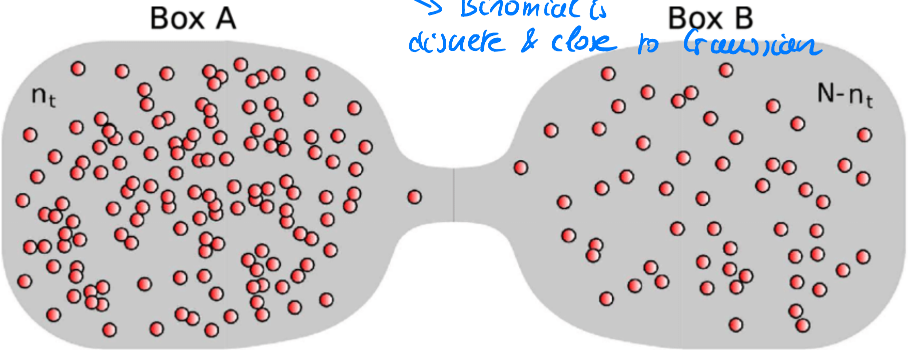
$$X_n \equiv \# \text{ molecules in Box } A \text{ at time } n$$

# Ehrenfest diffusion model (2)

Intuition : mean for inv. measure = $\frac{N}{2}$

$\Rightarrow$ inv. measure $\simeq \frac{N}{2}$ + "Gaussian" fluctuations

↳ Binomial is discrete & close to Gaussian



# Ehrenfest diffusion model (2)

**Proposition 45.**

For Ehrenfest's model,

1. $X$ is a Markov chain with
$$p_{i,i+1} = 1 - \frac{i}{\cancel{m}\,N}, \quad \text{and} \quad p_{i,i-1} = \frac{i}{\cancel{m}\,N}$$
2. $X$ is reversible in equilibrium with
$$\pi = \text{Bin}\left(N, \frac{1}{2}\right), \quad \text{that is} \quad \pi_i = \binom{N}{i}\left(\frac{1}{2}\right)^{\cancel{m}\,N}$$