Hyp X non-null persistent, irreducible
$X(0) \sim \pi$, where π inv dist

Object of interest : $Y_n = X_{N-n}$, where
N fixed and $0 \le n \le N$

Facts :

① Y is MC, with transition $\frac{\pi_j}{\pi_i} P_{ji}$

② we say that X is reversible if
Y has the same transition as X, ie

$$\frac{\pi_j}{\pi_i} P_{ji} = P_{ij} \iff \pi_i P_{ij} = \pi_j P_{ji}$$

Ehrenfest diffusion model (1)

Model: We consider

- Two boxes A and B
- Total of N gas molecules in $A \cup B$
- At time n, one molecule is picked from the N molecules
- This molecule changes box

Process: We set

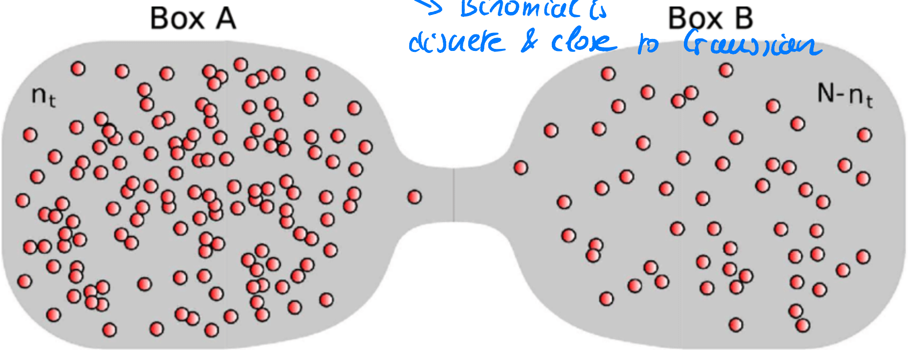
$$X_n \equiv \# \text{ molecules in Box } A \text{ at time } n$$

Ehrenfest diffusion model (2)

<u>Intuition</u>: mean for inv. measure $= \frac{N}{2}$

$\Rightarrow$ inv. measure $\simeq \frac{N}{2}$ + "Gaussian" fluctuations

↳ Binomial is discrete & close to Gaussian



Ehrenfest diffusion model (2)

Proposition 45.

For Ehrenfest's model,

1. X is a Markov chain with

$$p_{i,i+1} = 1 - \frac{i}{\cancel{m}\,N}, \quad \text{and} \quad p_{i,i-1} = \frac{i}{\cancel{m}\,N}$$

2. X is reversible in equilibrium with

$$\pi = \text{Bin}\left(N, \frac{1}{2}\right), \quad \text{that is} \quad \pi_i = \binom{N}{i} \left(\frac{1}{2}\right)^{\cancel{m}\,N}$$

strategy: prove $X_{n+1} = \varphi(X_n, Z_{n+1})$ with $Z_{n+1} \perp\!\!\!\perp$ past

X is a Markov chain we have

$$X_{n+1} = X_n + Z_{n+1}$$

$X_n = \#$ balls in A

where $Z_{n+1} \in \{-1, 1\}$

Z_{n+1} Rademacher r.v. with

$$p = P(Z_{n+1} = 1) = P(\text{ball picked in box B})$$

$$= \frac{N - X_n}{N}$$

$N - X_n$ = # balls in box B

$$= 1 - \frac{X_n}{N}$$

→ This a random prob. which depends on X_n (not $\perp\!\!\!\perp$ of past)

Side rmk If $z \sim \text{Rademacher}(p)$, how to simulate z?

we set $z = \mathbb{1}(U \leq p) - \mathbb{1}(U > p)$

with $U \sim \mathcal{U}([0,1])$ (only r.v. directly accessible on computer)

Then $\boxed{z \sim \text{Rademacher}(p)}$

Back to dynamics for X_n : we have seen

$$X_{n+1} = X_n + z_{n+1} \quad , \quad z_{n+1} = \text{Rademacher}\left(1 - \frac{X_n}{N}\right)$$

$$\Rightarrow X_{n+1} = X_n + \mathbb{1}\left(U_{n+1} \leq 1 - \frac{X_n}{N}\right) - \mathbb{1}\left(U_{n+1} > 1 - \frac{X_n}{N}\right)$$

$$\Rightarrow X_{n+1} = \varphi(X_n, U_{n+1}) \text{ , with } \{U_n ; n \geq 1\} \text{ iid } \mathcal{U}([0,1])$$

Partial conclusion

$$X_{n+1} = X_n + \mathbb{1}\left(U_{n+1} \leq 1 - \frac{X_n}{N}\right) - \mathbb{1}\left(U_{n+1} > 1 - \frac{X_n}{N}\right)$$

$\Rightarrow$ $\boxed{X \quad \text{Markov chain}}$

Transition : we are only interested in

$p_{i,i+1}$, $p_{i,i-1}$.

we have seen

$$\boxed{p_{i,i+1} =} \ \mathbb{P}\left(\varphi(i, U) = i+1\right) \quad , \text{with } U \sim \mathcal{U}([0,1])$$

$$= \mathbb{P}\left(i + \mathbb{1}\left(U \leq 1 - \frac{i}{N}\right) - \mathbb{1}\left(U > 1 - \frac{i}{N}\right) = i+1\right)$$

$$= \mathbb{P}\left(\mathbb{1}\left(U \leq 1 - \frac{i}{N}\right) = 1\right)$$

$$= \mathbb{P}\left(U \leq 1 - \frac{i}{N}\right) \boxed{= \quad 1 - \frac{i}{N}}$$

In the same way

$$p_{i,i-1} = \text{"prob to pick a ball in box } A\text{"}$$

$$= \frac{i}{N}$$

Next claim : $\pi = \mathrm{Bin}(N, \frac{1}{2})$ inv. distribution.

For this it is enough to prove

$$\pi_i \, p_{ij} = \pi_j \, p_{ji}$$

Here we will just check

$$\boxed{\pi_i \quad p_{i,i+1} \quad = \quad \pi_{i+1} \quad p_{i+1,i}}$$

($\pi_i \quad p_{i,i-1} \quad = \quad \pi_{i-1} \quad p_{i-1,i}$ to be checked)

$$\pi_i = [\mathrm{Bin}(N, \tfrac{1}{2})]_i = \binom{N}{i} \left(\tfrac{1}{2}\right)^N$$

Verifying

$$\pi_i \quad p_{i,i+1} = \pi_{i+1} \quad p_{i+1,i}$$

$$\boxed{\pi_i \quad p_{i,i+1}} = \frac{N!}{i!\,(N-i)!} \frac{1}{2^N} \overbrace{\left(1 - \frac{i}{N}\right)}^{\frac{(N-i)}{N}}$$

$$= \frac{(N-1)!}{i!\,(N-i-1)!} \frac{1}{2^N}$$

$$\boxed{\pi_{i+1} \quad p_{i+1,i}} = \frac{N!}{(i+1)!\,(N-i-1)!} \frac{1}{2^N} \left(\frac{i+1}{N}\right)$$

$$= \frac{(N-1)!}{i!\,(N-i-1)!} \frac{1}{2^N}$$

Thus π is reversible in equilibrium

Proof of Proposition 45 (1)

Markov chain: One can write

$$X_{n+1} = X_n - (2Y_{n+1} - 1), \quad \text{where} \quad Y_{n+1} \sim \mathcal{B}\left(\frac{X_n}{N}\right)$$

Otherwise stated: We also have

$$X_{n+1} = X_n - \mathbf{1}_{\left(U_{n+1} \le \frac{X_n}{N}\right)} + \mathbf{1}_{\left(U_{n+1} > \frac{X_n}{N}\right)} \equiv \varphi(X_n, U_{n+1}),$$

where $\{U_k;\, k \ge 1\}$ are i.i.d $\mathcal{U}([0,1])$

Conclusion: X is a Markov chain with

$$p_{i,i+1} = 1 - \frac{i}{m}, \quad \text{and} \quad p_{i,i-1} = \frac{i}{m}$$

Proof of Proposition 45 (2)

Reversible in equilibrium: One checks that

$$\begin{aligned} \pi_i \, p_{i,i+1} &= \pi_{i+1} \, p_{i+1,i} \\ \pi_i \, p_{i,i-1} &= \pi_{i-1} \, p_{i-1,i} \end{aligned}$$