Outline

Irreducible case

Theorem 46.

Let

- X irreducible Markov chain with transition P
- S finite

Then:

$$X \text{ is non-null persistent}$$

Perron-Frobenius theorem

Theorem 47.

Let

- X irreducible Markov chain with transition P
- S finite with $|S| = N$, X has period d

Then:

$P\mathbb{1} = \mathbb{1}$

1. $\lambda_1 = 1$ is an eigenvalue of P
2. Let $\omega = e^{\frac{2\pi\imath}{d}}$. Then the following are eigenvalues of P:

$$\lambda_k = \omega^{k-1}, \quad \text{for} \quad k = 1, \ldots, d$$

3. Remaining eigenvalues:

$$\lambda_{d+1}, \ldots, \lambda_N, \quad \text{with} \quad |\lambda_j| < 1$$

Large time behavior

Theorem 48.

Let

- X irreducible Markov chain with transition P
- S finite with $|S| = N$
- $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_N)$ eigenvalue matrix
- Hyp: Eigenvalues λ_j all distinct
- $V = [v_1, \ldots, v_n]$ eigenvector matrix

Then:

$\Lambda^n = \mathrm{Diag}(\lambda_1^n, \ldots, \lambda_N^n)$

1. $P^n = V \Lambda^n V^{-1}$

$n \to \infty$

due to the fact that $\lambda_1^n = 1^n = 1$

$\lambda_j^n \longrightarrow 0$ if $j > 1$

2. If X is aperiodic we have

$$\lim_{n\to\infty} P^n = V \, \mathrm{Diag}(1, 0, \ldots, 0) V^{-1}$$

Inbreeding model (1)

Model:

- Spinach population
- Genetic information contained in chromosomes
- 6+6 identical pairs of chromosomes
- Sites $C_1, \ldots, C_M$ for chromosomes
  $\hookrightarrow$ We just look at C_1 for 1 chromosome
- $C_1 \in \{a, A\}$ for each pair
- Types: given by $S = \{AA, aA, aa\}$
- $X_n \equiv$ Value of type at generation n for a typical spinach
- Self reproduction model with meiosis
  $\hookrightarrow$ shuffle of C_i's between pairs $\longrightarrow$ take 2 pairs and shuffle their a A

Mechanism

$AA \times AA \longrightarrow AA$ always

$aA \times aA \longrightarrow \underbrace{aa}_{p=1/4} \quad \underbrace{aA}_{p=1/2} \quad \underbrace{AA}_{p=1/4}$

$aa \times aa \longrightarrow aa$ with $p=1$

X_n is a MC

Strategy: $X_{n+1} = \varphi(X_n, Y_{n+1})$

Here

$$X_{n+1} = \underbrace{(AA)\,\mathbb{1}_{(X_n=(AA))} + (aa)\,\mathbb{1}_{(X_n=(aa))} + Y_{n+1}\,\mathbb{1}_{(X_n=(aA))}}_{=\varphi(X_n, Y_{n+1})}$$

where $\{Y_n ; n \geq 1\}$ iid with $\begin{cases} P(Y_{n+1} = AA) = \frac{1}{4} \\ P(Y_{n+1} = aA) = \frac{1}{2} \\ P(Y_{n+1} = aa) = \frac{1}{4} \end{cases}$

Transition

$$p_{ij} = P(\varphi(i, Y_1) = j)$$

We get, on $S = \{AA, aA, aa\}$

$$P = \begin{pmatrix} 1 & \boxed{0} & 0 \\ 1/4 & \boxed{1/2} & 1/4 \\ 0 & \boxed{0} & 1 \end{pmatrix}$$

$$P \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Graph

$$\begin{tikzcd} AA & \\ & aA \arrow[ul] \arrow[dl] \\ aa & \end{tikzcd}$$

Recurrent (absorbant) classes:

$\{AA\}$, $\{aa\}$

Transient element:

aA

Eigenvalues : roots of

$$\det(P-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1/4 & \frac{1}{2}-\lambda & 1/4 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left(\frac{1}{2}-\lambda\right)(1-\lambda)$$

we get $\boxed{\lambda_1 = 1 \quad , \quad \lambda_2 = 1 \quad , \quad \lambda_3 = \frac{1}{2}}$

Eigenvectors : solutions to $(P-\lambda I)v = 0$. we get

$$\boxed{V = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} \qquad V^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

Inbreeding model (2)

Transition rules: If all shuffles are equally likely we get

- $AA \times AA \longrightarrow AA$, with $p = 1$
- $aA \times aA \longrightarrow aa$ with $p = \frac{1}{4}$, aA with $p = \frac{1}{2}$, AA with $p = \frac{1}{4}$
- $aa \times aa \longrightarrow aa$, with $p = 1$

Transition matrix: We get

$$P = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

Inbreeding model (3)

Classification of states: With the graph we find

- aa and AA are persistent
- aA is transient

Eigenvalues: We find

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = \frac{1}{2}$$

Eigenvectors: We get

$$V = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad V^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{pmatrix}$$

Inbreeding model (4)

Large time behavior: We get

$$
\begin{aligned}
P^n &= V^{\cancel{-1}} \overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \left(\frac{1}{2}\right)^n \end{pmatrix}}^{\Lambda^n} V^{-1} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} & \left(\frac{1}{2}\right)^n & \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1} \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Limiting behavior: We have

$$
\lim_{n\to\infty} P^n = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}
$$

Additional information:
geometric
convergence