Outline

A model for radioactive particles emission

Model for the process

on [0,t]

- $N(t) \equiv \#$ particles emitted ~~at~~ until time t
- $N = \{N(t);\, t \geq 0\}$
- $N(0) = 0$ and $N(t) \in \mathbb{N}$
- $N(s) \leq N(t)$ if $s \leq t$

Emission model:

h small parameter , we will often take $h \to 0$

- In $(t, t+h)$ there might/might not be emissions
- h small $\Longrightarrow$ likelihood of emission is $\simeq \lambda h$
  $\hookrightarrow$ with an intensity λ
- At most 1 emission if h is small

Definition of Poisson process

$g(h) = o(h)$ if $\lim_{h \to 0} \frac{g(h)}{h} = 0$

Ex: $g(h) = h^2 = o(h)$

Definition 1.

Let

- $N = \{N(t);\ t \geq 0\}$ process with $N(0) = 0$ and $N(t) \in \mathbb{N}$

Then N is a Poisson process if

- $N(0) = 0$ and $t \mapsto N(t)$ is $\nearrow$
- Probability $\mathbf{P}\left(N(t+h) = n + m \mid N(t) = n\right)$ of the form

$n \in \mathbb{N}$

$\lambda h + g_1(h)$

$$\begin{cases} \lambda h + o(h) & \text{if } m = 1 \\ o(h) \; g_2(h) & \text{if } m > 1 \\ 1 - \lambda h + \underbrace{o(h)}_{g_3(h)} & \text{if } m = 0 \end{cases}$$

$$\frac{|g_1(h)| + |g_2(h)| + |g_3(h)|}{h} \to 0$$

- $t \geq s$: $N(t) - N(s) \perp\!\!\!\perp$ emissions on $[0, s]$

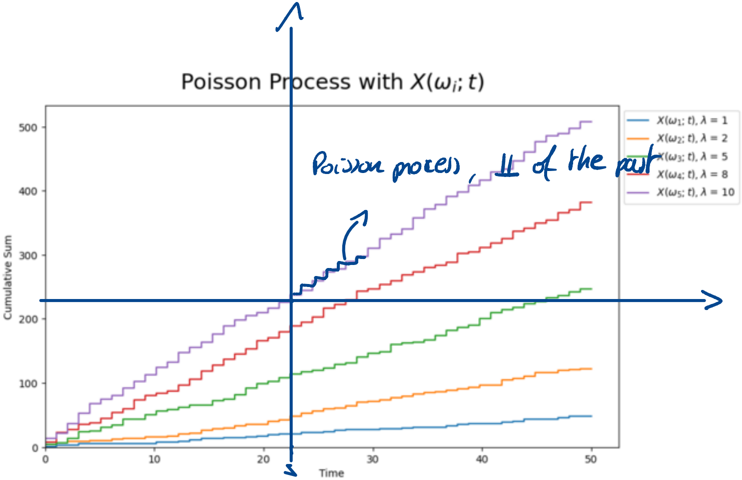
Rmk If N Poisson, h small

$$\mathbb{P}(N(t+h) = n \mid N(t)=n)$$
$$+ \mathbb{P}(N(t+h) = n+1 \mid N(t)=n)$$
$$= \underbrace{1 + o(h)}_{g(h)} \qquad \left(Ex: \quad g(h) = 1 - \frac{h^2}{16} \right)$$

Rmk If $m > 1$,

$$\underbrace{\mathbb{P}(N(t+h)=n \mid N(t)=n)}_{1-\lambda h + o(h)} \gg \overbrace{\mathbb{P}(N(t+h)=n+1 \mid N(t)=n)}^{\lambda h + o(h)}$$
$$\gg \underbrace{\mathbb{P}(N(t+h) = n+m \mid N(t)=n)}_{o(h)}$$

Paths of a Poisson process



Vocabulary

Terminology for Poisson processes:

- $N(t)$ is interpreted as a number of arrivals
- N is called counting process

Broader context:

- N is a simple example of continuous time Markov chain
- More general objects: in next section

Birth of Poisson process

3 independent discoveries:

- Lund, Sweden, 1903
  $\hookrightarrow$ Actuarial studies
- Erlang, Denmark, 1909
  $\hookrightarrow$ Telecommunication networks
- Rutherford, New Zealand, 1910
  $\hookrightarrow$ Particle emission

Marginal distribution

Theorem 2.

Let

- N Poisson process with intensity λ
- $t \geq 0$

In particular, $\mathbb{E}[N(t)] = \lambda t$

Then

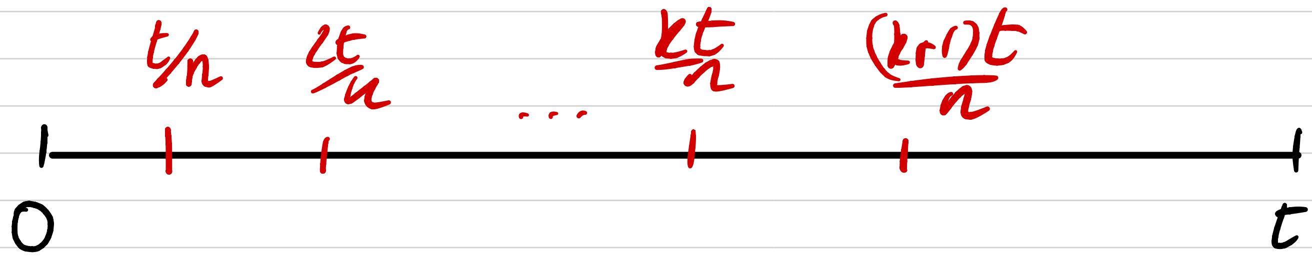
$$N(t) \sim \mathcal{P}(\lambda t),$$

that is for $j \in \mathbb{N}$ we have

$$\mathbf{P}(N(t) = j) = \frac{(\lambda t)^j}{j!} e^{-\lambda t}$$

Intuition about $N(t) \sim \mathcal{P}(\lambda t)$

$h = \frac{t}{n}$



On each interval, let

$$X_k = \mathbb{1}\left(\text{particle observed on } \left[\tfrac{kt}{n}, \tfrac{(k+1)t}{n}\right)\right)$$

average # of part. on $[0,t)$ $= \lambda t$

According to the specifications

$$X_k \approx \mathcal{B}\left(\lambda \frac{t}{n}\right), \quad \text{and } X_k \text{ indep}$$

$$N(t) \simeq \sum_{k=0}^{n-1} X_k = \mathrm{Bin}\left(n, \frac{\lambda t}{n}\right) \xrightarrow{n \to \infty} \boxed{\mathcal{P}(\lambda t)}$$

$$\simeq N\left(\frac{(k+1)t}{n}\right) - N\left(\frac{kt}{n}\right)$$