Probability generating functions

Definition 4.

Let

- X random variable with values in $\mathbb{Z}$
- f_X pmf of X

$$= \sum s^i \, P(X=i)$$

We set

$$G_X(s) = \mathbf{E}\left[s^X\right] = G_{f_X}(s)$$

$$E[\varphi(X)] = \sum_i \varphi(i) \, P(X=i) = \sum_i \varphi(i) f_X(i)$$

Transforms for random variables

(i) If $X \in \mathbb{Z}$ we usually consider

$$G_X(s) = E[s^X] \rightarrow \text{entire series}$$

(ii) If $X \in \mathbb{R}_+$, we consider Laplace transform

$$\phi(\lambda) = E[e^{-\lambda X}] \rightarrow \text{monotonicity}$$

(iii) If $X \in \mathbb{R}$ we consider characteristic fct

$$\phi(\xi) = E[\underbrace{e^{i \xi X}}_{\text{bounded}}] \quad \text{(Fourier transform)}$$

Properties of the generating function (1)

For a pmf, $G_X(1) = \sum_i f_X(i)\, 1^i = \sum_i f_X(i) = 1$

$R \geq 1$ if we are dealing with a pmf

Some properties:

1. Convergence: There exists $R \geq 0$ such that $G_X(s)$
   - Converges absolutely if $|s| < R$
   - Diverges if $|s| > R$
   - The sum is uniformly convergent on $\{s;\ |s| < R'\}$ if $R' < R$

     $\hookrightarrow \sum_i f_X(i)\, s^i$
2. Differentiation: One can differentiate term by term at s
   $\hookrightarrow$ such that $|s| < R$

   $\hookrightarrow$ $G_X(s) = \sum_i f_X(i)\, s^i$

   $G'_X(s) = \sum_i i\, f_X(i)\, s^{i-1}$

Properties of the generating function (2)

Some more properties:

3. Uniqueness: Assume
   - $G_a(s) = G_b(s)$ for $|s| < R' \leq R$ → with $R' > 0$

   Then
   $$(a_n)_{n\geq 0} = (b_n)_{n\geq 0}, \quad \text{and} \quad a_n = \frac{1}{n!} G_a^{(n)}(0)$$

4. Abel theorem: Assume
   - $a_i \geq 0$
   - $G_a(s) < \infty$ for $|s| < 1$

   Then
   $$\lim_{s \nearrow 1} G_a(s) = \sum_{i=0}^{\infty} a_i$$

Use of Abel: For some r.v T we have $P(T=\infty) > 0$

In that case

$G_T(s) = \sum_i P(T=i)\, s^i$

$s \to 1$

$\sum_i P(T=i) = P(T<\infty)$

Bernoulli random variables

$X \sim B(p) \qquad p \in (0,1)$

$P(X=1) = p \qquad P(X=0) = 1-p$

State space : $\{0,1\}$

$$G_X(s) = \sum_{i \in \{0,1\}} s^i \, P(X=i)$$

$$= s^0 (1-p) + s^1 p$$

$$\boxed{G_X(s) = (1-p) + sp}$$

Bernoulli random variable (1)

Notation:

$$X \sim \mathcal{B}(p) \text{ with } p \in (0,1)$$

State space:

$$\{0,1\}$$

Pmf:

$$\mathbf{P}(X=0) = 1-p, \quad \mathbf{P}(X=1) = p$$

Expected value, variance, generating function:

$$\mathbf{E}[X] = p, \qquad \mathbf{Var}(X) = p(1-p), \qquad G_X(s) = (1-p) + p\,s$$

Bernoulli random variable (2)

17th
18th
19th

Use 1, success in a binary game:

- Example 1: coin tossing
  - $X = 1$ if H, $X = 0$ if T
  - We get $X \sim \mathcal{B}(1/2)$
- Example 2: dice rolling
  - $X = 1$ if outcome $= 3$, $X = 0$ otherwise
  - We get $X \sim \mathcal{B}(1/6)$

Use 2, answer yes/no in a poll

- $X = 1$ if a person feels optimistic about the future
- $X = 0$ otherwise
- We get $X \sim \mathcal{B}(p)$, with unknown p

Jacob Bernoulli

Some facts about Bernoulli:

- Lifespan: 1654-1705, in Switzerland
- Discovers constant e
- Establishes divergence of $\sum \frac{1}{n}$
- Contributions in diff. eq
- First law of large numbers
- Bernoulli:
  family of 8 prominent mathematicians
- Fierce math fights between brothers

Geometric r.v.

$X \sim G(p)$ state space: $\{1, 2, \dots\}$

$P(X = k) = p(1-p)^{k-1}$

$s^{k-1} s$

$G_X(s) = \sum_{k=1}^{\infty} p(1-p)^{k-1} s^k$

$= ps \sum_{k=1}^{\infty} (1-p)^{k-1} s^{k-1}$ ↗ geom series

$$\boxed{G_X(s) = \frac{ps}{1 - s(1-p)}}$$

Geometric random variable

Notation:

$$X \sim \mathcal{G}(p), \qquad \text{for } p \in (0,1)$$

State space:

$$E = \mathbb{N} = \{1, 2, 3, \dots\}$$

Pmf:

$$\mathbf{P}(X = k) = p\,(1-p)^{k-1}, \quad k \geq 1$$

Expected value, variance and generating function:

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}, \qquad G_X(s) = \frac{p\,s}{1 - s(1-p)}$$

Geometric random variable (2)

Use:

- Independent trials, with $\mathbf{P}(\text{success}) = p$
- $X = \#$ trials until first success

Example: dice rolling

- Set $X = $ 1st roll for which outcome $= 6$
- We have $X \sim \mathcal{G}(1/6)$

Computing some probabilities for the example:

$$\begin{aligned} \mathbf{P}(X = 5) &= \left(\frac{5}{6}\right)^4 \frac{1}{6} \simeq 0.08 \\ \mathbf{P}(X \geq 7) &= \left(\frac{5}{6}\right)^6 \simeq 0.33 \end{aligned}$$

Geometric random variable (3)

Computation of $\mathbf{E}[X]$: Set $q = 1 - p$. Then

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{i=1}^{\infty} i q^{i-1} p \quad = \sum_{i=1}^{\infty} i \; P(X=i) \\
&= \sum_{i=1}^{\infty} (i-1) q^{i-1} p + \sum_{i=1}^{\infty} q^{i-1} p \\
&= q \, \mathbf{E}[X] + 1
\end{aligned}
$$

Conclusion:

$$\mathbf{E}[X] = \frac{1}{p}$$

Generating function and moments

Theorem 5.

Let X be a random variable with generating function G_X. Then

1. $\mathbf{E}[X] = G'(1)$
2. $\mathbf{E}[X(X-1)\cdots(X-k+1)] = G_X^{(k)}(1)$

Remark: If the radius of convergence for G_X is 1, then

$$G_X^{(k)}(1) = \lim_{s \nearrow 1} G_X^{(k)}(s)$$

$E[X]$ from G_X for $X \sim g(p)$

We have seen that $G_X(s) = \dfrac{ps}{1 - s(1-p)}$

$$G_X(s) = \frac{ps}{-(1-p)s + 1}$$

Thus $G_X'(s) = \dfrac{p}{(1-(1-p)s)^2} = \dfrac{p}{(1-(1-p)s)^2}$

and $\boxed{G_X'(1) = E[X] = \dfrac{1}{1-(1-p)} = \dfrac{1}{p}}$

We also have $G_X''(s) = \dfrac{2p(1-p)}{(1-(1-p)s)^3}$ (check)

Thus $E[X(X-1)] = G_X''(1) = \dfrac{2p(1-p)}{p^3} = \dfrac{2(1-p)}{p^2}$

↳ then compute $V(X)$

Computing moments with generating functions

Situation: Consider $p \in (0,1)$ and

$$X \sim \mathcal{G}(p)$$

Derivatives of G_X: We find

$$\begin{aligned} G_X'(s) &= \frac{p}{(1-(1-p)s)^2} \\ G_X''(s) &= \frac{2p(1-p)}{(1-(1-p)s)^3} \end{aligned}$$

Moments: We get

$$\mathbf{E}[X] = \frac{1}{p}, \qquad \mathbf{Var}(X) = \frac{1-p}{p^2}$$