$\boxed{\text{Summary on Poisson process}}$. $N(t)$ is PP if

(i) $N(t) \in \mathbb{N}$, $N(0) = 0$

(ii) $t \mapsto N(t)$ is $\nearrow$

(iii) $\mathbb{P}(N(t+h) = n+m \mid N(t) = n)$

$$= \begin{cases} 1 - \lambda h + g_1(h) & \text{if } m = 0 \\ \lambda h + g_2(h) & \text{if } m = 1 \\ g_3(h) & \text{if } m > 1 \end{cases}$$

(iv) $N(t) - N(s) \perp\!\!\!\perp$ "past"

with $\lim_{h \to 0} \dfrac{|g_1(h)| + |g_2(h)| + |g_3(h)|}{h} \to 0$

Notation: we say $g_1(h), g_2(h), g_3(h) = o(h)$

First important result. For all $t \geq 0$

$$N(t) \sim P(\lambda t)$$

Intuition This stems from

$$\text{Bin}\left(n, \frac{\mu}{n}\right) \xrightarrow{n \to \infty} P(\mu)$$

means convergence in distribution

Proof of $\text{Bin}(n, \frac{\mu}{n})$ " $\longrightarrow$ " $\mathcal{P}(\mu)$

It boils down to prove, for $k \in \mathbb{N}$, that

$$\mathbb{P}(X_n = k) \longrightarrow \mathbb{P}(X = k)$$

where $X_n \sim \text{Bin}(n, \frac{\mu}{n})$, $X \sim \mathcal{P}(\mu)$

Computation

$$\mathbb{P}(X_n = k) = \binom{n}{k} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} \frac{\mu^k}{n^k} \left(1 - \frac{\mu}{n}\right)^{n-k}$$

$$= \frac{\mu^k}{k!} \frac{n!}{(n-k)!\, n^k} \left(1 - \frac{\mu}{n}\right)^n \left(1 - \frac{\mu}{n}\right)^{-k}$$

$$\mathbb{P}(X_n = k) = \frac{\mu^k}{k!} \overbrace{\frac{n!}{(n-k)!\, n^k}}^{A(n)} \overbrace{\left(1 - \frac{\mu}{n}\right)^n}^{B(n)} \overbrace{\left(1 - \frac{\mu}{n}\right)^{-k}}^{C(n)}$$

Then

$$A(n) = \frac{n(n-1)\cdots(n-k+1)}{n^k} = \frac{n^k + \overbrace{P_{k-1}(n)}^{\text{polynomial degree } k-1}}{n^k} \xrightarrow{n \to \infty} 1$$

$$B(n) = e^{n \overbrace{\ln\left(1 - \frac{\mu}{n}\right)}^{\sim -\mu/n}} \xrightarrow{n \to \infty} e^{-\mu}$$

$$C(n) = \frac{1}{\left(1 - \frac{\mu}{n}\right)^k} \xrightarrow{n \to \infty} 1 \quad (\text{cont. function})$$

<u>conclusion</u>

$$\boxed{\mathbb{P}(X_n = k) \longrightarrow \frac{\mu^k}{k!} e^{-\mu} = \mathbb{P}(X = k)}$$

no more than 1 arrival in small time

Proof of Thm 2

$N(t+h) \geq N(t)$

$$\mathbb{P}(N(t+h) = j)$$

$$= \sum_{i=0}^{j} \mathbb{P}(N(t+h) = j \mid N(t) = i)\ \mathbb{P}(N(t) = i)$$

h small

$$\approx \mathbb{P}(N(t+h) = j \mid N(t) = j)\ \mathbb{P}(N(t) = j)$$

$$+ \mathbb{P}(N(t+h) = j \mid N(t) = j-1)\ \mathbb{P}(N(t) = j-1) + o(h)$$

$$= (1 - \lambda h + o(h))\ \mathbb{P}(N(t) = j)$$

$$+ (\lambda h + o(h))\ \mathbb{P}(N(t) = j-1) + o(h)$$

Notation: we set $p_j(t) = \mathbb{P}(N(t) = j)$.

we get

$$\boxed{p_j(t+h) = (1 - \lambda h)\ p_j(t) + \lambda h\ p_{j-1}(t) + o(h)}$$

$p_j(t) = \mathbb{P}(N(t)=j)$

We have found

$$p_j(t+h) = (1-\lambda h)\, p_j(t) + \lambda h\, p_{j-1}(t) + o(h)$$

$$\Leftrightarrow \quad p_j(t+h) - p_j(t) = -\lambda h\, p_j(t) + \lambda h\, p_{j-1}(t) + o(h)$$

$$\Leftrightarrow \quad \frac{1}{h}\left(p_j(t+h) - p_j(t)\right) = -\lambda\, p_j(t) + \lambda\, p_{j-1}(t) + o(1)$$

Take $h \to 0$. We get

$$p_j'(t) = -\lambda\, p_j(t) + \lambda\, p_{j-1}(t)$$

We get an ∞ (in j) system of linear differential equations

2 strategies : Solve by induction

Generating functions

$= \mathbb{P}(N(0)=0)$

System

$$p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t)$$

Case $j=0$ Since $p_{-1}(t)=0$, the eq is

$$p_0'(t) = -\lambda p_0(t) \quad , \quad p_0(0) = 1$$

$$\Leftrightarrow \quad \frac{p_0'(t)}{p_0(t)} = -\lambda \quad \text{(separable)}$$

$$\Rightarrow \quad \ln(p_0(t)) = -\lambda t + c_1$$

$$p_0(t) = c_2 e^{-\lambda t}$$

with $p_0(0)=1$, we get

$X \sim \mathcal{P}(\lambda t)$

$$\boxed{p_0(t) = e^{-\lambda t} = \mathbb{P}(X=0)}$$

$$p_j'(t) = -\lambda p_j(t) + \lambda p_{j-1}(t)$$

Diff for $G_t(s) = \sum_{j=0}^{\infty} p_j(t)\, s^j$. we have

$$\partial_t G_t(s) = p_0'(t) + \sum_{j=1}^{\infty} p_j'(t)\, s^j$$

$$= -\lambda p_0(t) + \sum_{j=1}^{\infty} \left(-\lambda p_j(t) + \lambda p_{j-1}(t)\right) s^j$$

$$= -\lambda p_0(t) + \sum_{j=1}^{\infty} -\lambda p_j(t)\, s^j$$

$$+ s \sum_{j=1}^{\infty} \lambda p_{j-1}(t)\, s^{j-1}$$

$$= -\lambda G_t(s) + \lambda s\, G_t(s)$$

$$\Rightarrow \quad \partial_t G_t(s) = \lambda(s-1)\, G_t(s), \quad G_0(s) = 1$$

$$\Rightarrow \quad \boxed{G_t(s) = e^{\lambda(s-1)t}} \quad \Rightarrow \quad p_j(t) \text{ is } \mathcal{P}(\lambda t)$$

Proof of Theorem 2 (1)

Conditioning on a small interval: We have

$$
\begin{aligned}
&\mathbf{P}\left(N(t+h)=j\right)\\
&=\sum_{i\in S}\mathbf{P}\left(N(t+h)=j \,|\, N(t)=i\right)\mathbf{P}\left(N(t)=i\right)\\
&=\sum_{i\in S}\mathbf{P}\left((j-i)\text{ arrivals in }(t,t+h]\right)\mathbf{P}\left(N(t)=i\right)\\
&=\mathbf{P}\left(\text{no arrivals in }(t,t+h]\right)\mathbf{P}\left(N(t)=j\right)\\
&\quad+\mathbf{P}\left(\text{one arrival in }(t,t+h]\right)\mathbf{P}\left(N(t)=j-1\right)+o(h)\\
&=(1-\lambda h)\,\mathbf{P}\left(N(t)=j\right)+\lambda h\,\mathbf{P}\left(N(t)=j-1\right)+o(h)
\end{aligned}
$$

Proof of Theorem 2 (2)

Probability as a function: We set

$$p_j(t) = \mathbf{P}(N(t) = j)$$

Equation on small intervals: We have seen

$$\begin{aligned} p_0(t+h) &= (1-\lambda h)\, p_0(t) + o(h) \\ p_j(t+h) &= \lambda h\, p_{j-1}(t) + (1-\lambda h)\, p_j(t) + o(h) \end{aligned}$$

Equivalent form with differences:

$$\begin{aligned} p_0(t+h) - p_0(t) &= -\lambda h\, p_0(t) + o(h) \\ p_j(t+h) - p_j(t) &= \lambda h\, (p_{j-1}(t) - p_j(t)) + o(h) \end{aligned}$$

Proof of Theorem 2 (3)

Recall:

$$
\begin{aligned}
p_0(t+h) - p_0(t) &= -\lambda h\, p_0(t) + o(h) \\
p_j(t+h) - p_j(t) &= \lambda h\,(p_{j-1}(t) - p_j(t)) + o(h)
\end{aligned}
$$

Differentiating: We end up with a system of ode's

$$
\begin{aligned}
p_0'(t) &= -\lambda\, p_0(t) \\
p_j'(t+h) &= \lambda\, p_{j-1}(t) - \lambda\, p_j(t)
\end{aligned}
$$

Initial condition:

$$
p_j(0) = \delta_{j0} \equiv \mathbf{1}_{(j=0)}
$$

Proof of Theorem 2 (4)

Recall: We have obtained a system of ode's

$$\begin{aligned} p_0'(t) &= -\lambda\, p_0(t) \\ p_j'(t+h) &= \lambda\, p_{j-1}(t) - \lambda\, p_j(t) \end{aligned}$$

A family of generating functions: We set

$$G_t(s) = \mathbf{E}\left[s^{N(t)}\right] = \sum_{j=0}^{\infty} p_j(t) s^j$$

Strategy: From the system of ode's
$\hookrightarrow$ deduce a single ode for $t \mapsto G_t(s)$

Proof of Theorem 2 (5)

Differential equation for G: We have

$$
\begin{aligned}
\frac{\partial G_t(s)}{\partial t} &= \sum_{j=0}^{\infty} p_j'(t) s^j \\
&= -\lambda\, p_0(t) + \sum_{j=1}^{\infty} \left(\lambda\, p_{j-1}(t) - \lambda\, p_j(t)\right) s^j \\
&= -\lambda\, G_t(s) + \lambda s \sum_{j=1}^{\infty} p_{j-1}(t) s^{j-1} \\
&= -\lambda\, G_t(s) + \lambda s G_t(s) \\
&= \lambda(s-1) G_t(s)
\end{aligned}
$$

Proof of Theorem 2 (6)

Recall: $u_t \equiv G_t(s)$ verifies

$$u' = \lambda(s-1)u, \qquad u_0 = 1$$

Expression for $G_t(s)$: We find

$$G_t(s) = \exp\left(\lambda(s-1)t\right)$$

Conclusion:

$$N(t) \sim \mathcal{P}(\lambda t)\,,$$

Relation with binomial random variables

Another way to prove $N(t) \sim \mathcal{P}(\lambda t)$:

1. Partition $[0, t]$ in subintervals $[(\ell - 1)h, \ell h]$
2. On each subinterval, set $Z_\ell = \mathbf{1}_{(\text{arrival in } [(\ell-1)h, \ell h])}$
3. We have that $\{Z_\ell;\ \ell \geq 1\}$ is i.i.d with common law $\mathcal{B}(\lambda h)$
4. We have $N(t) \simeq \sum_{\ell=1}^{t/h} Z_\ell$, thus

$$N(t) \simeq \text{Bin}\left(\frac{t}{h};\ \lambda h\right) \xrightarrow{h \to 0} \mathcal{P}(\lambda t)$$