Inter-arrival times

Definition 3.

Let

- N Poisson process with intensity λ

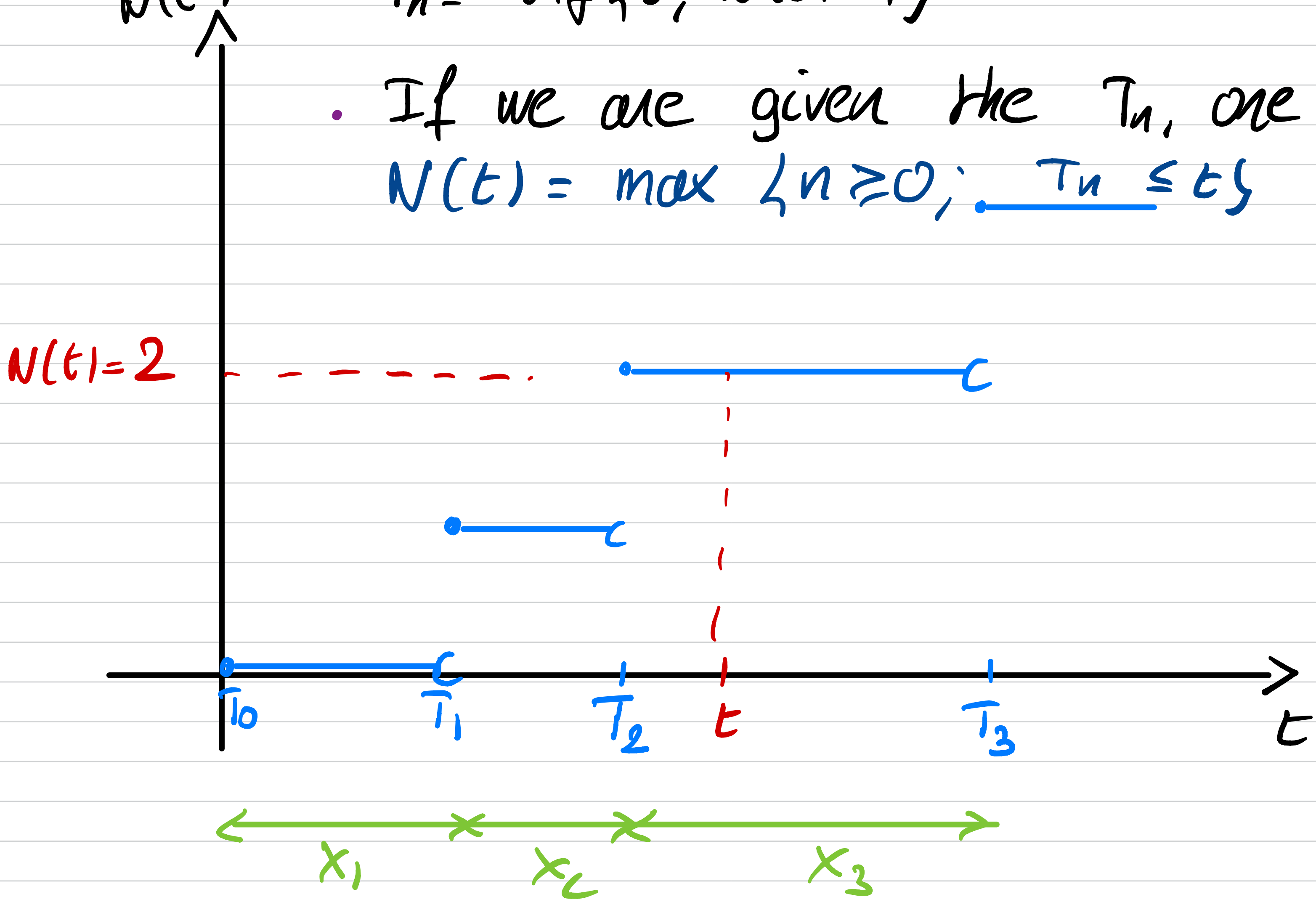
We define $T_0 = 0$ and

$$\begin{aligned} T_n &= \inf\{t \geq 0;\, N(t) = n\} \\ X_n &= T_n - T_{n-1} \end{aligned}$$

Then X_n is called inter-arrival time

- If we are given $N(t)$, one can deduce the T_n's:
$T_n = \inf\{t;\ N(t) = n\}$

- If we are given the T_n, one can deduce N
$N(t) = \max\{n \geq 0;\ T_n \leq t\}$



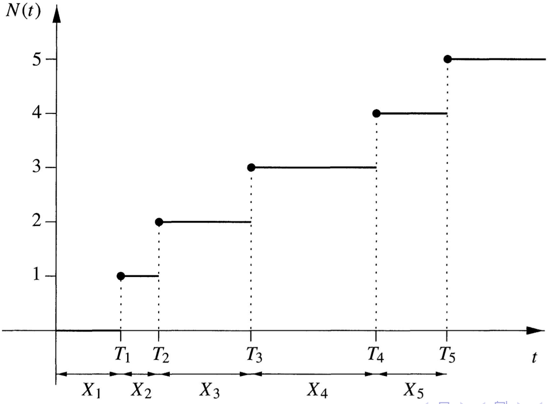
Another remark: $T_n = \sum_{j=1}^{n} X_j$

From X to N

N as a function of X: We have

$$T_n = \sum_{i=1}^{n} X_i$$

$$N(t) = \max\left\{n \geq 0;\ T_n \leq t\right\}$$



Distribution of the inter-arrival times

Theorem 4.

Let

- N Poisson process with intensity λ
- $\{X_j;\, j \geq 1\}$ inter-arrival times

Then

The X_j's are i.i.d with common distribution $\mathcal{E}(\lambda)$

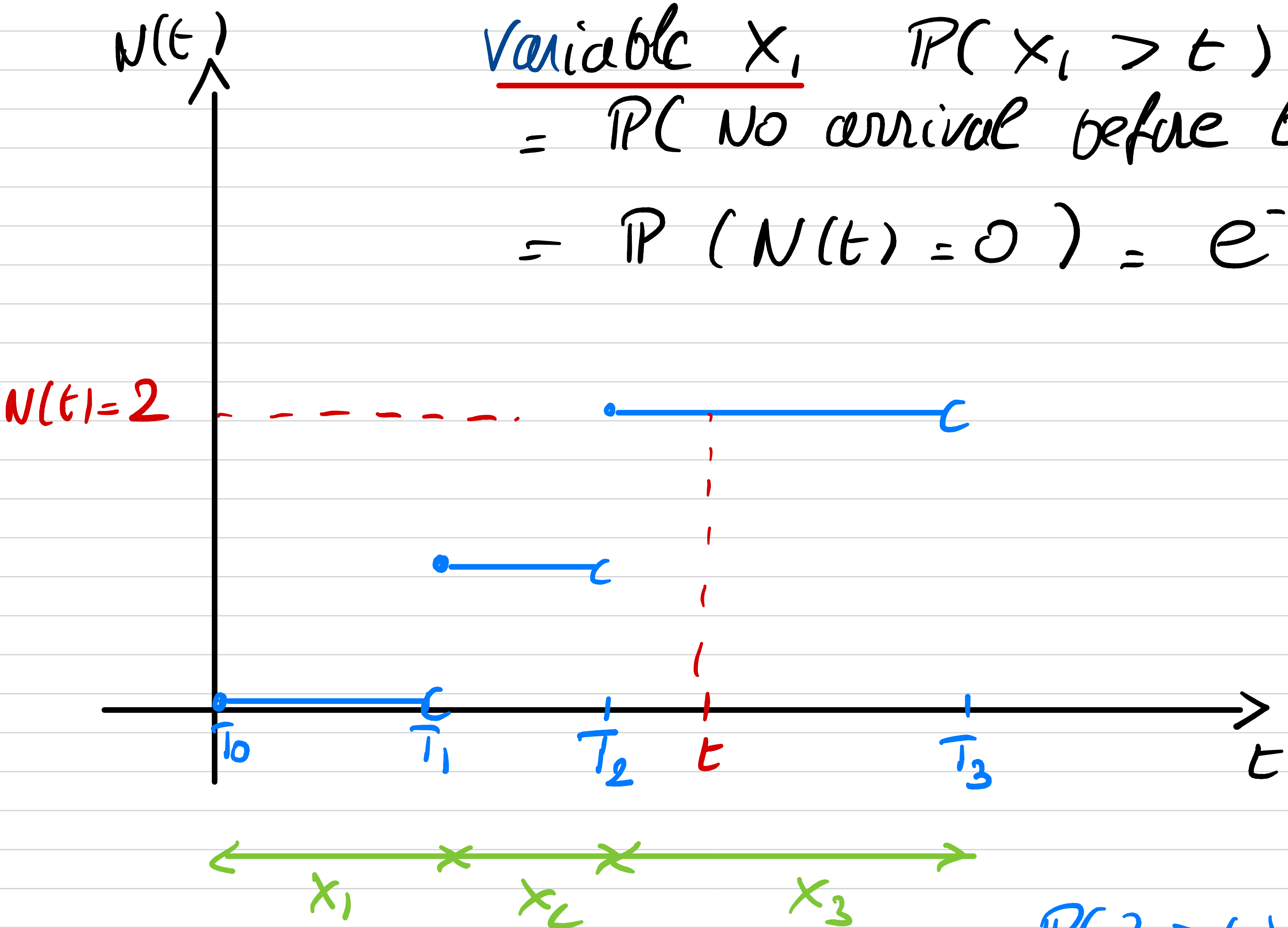
$E[X_j] = \frac{1}{\lambda}$

If λ (intensity) is large, we will not wait too long until the next arrival

Variable X_1

$$P(X_1 > t) = P(\text{No arrival before } t)$$

$$= P(N(t) = 0) = e^{-\lambda t}\frac{(\lambda t)^0}{0!}$$



$= P(Z > t)$ with $Z \sim \mathcal{E}(\lambda)$

We get $\forall t \geq 0$

$$P(X_1 > t) = e^{-\lambda t} = \int_t^\infty \lambda e^{-\lambda x}\,dx$$

Variable X_2

$? \; /\!/ \; e^{-\lambda t} \Rightarrow \begin{cases} (i) & X_2 \sim \mathcal{E}(\lambda) \\ (ii) & X_2 \perp\!\!\!\perp X_1 \end{cases}$

$$\mathbb{P}(X_2 > t \mid X_1 = t_1)$$

no arrival in $(t_1, t_1 + t]$

$$= \mathbb{P}\big((N(t_1+t) - N(t_1)) = 0 \mid X_1 = t_1, N(t_1) = 1\big)$$

Since $N(t_1+t) - N(t_1) \perp\!\!\!\perp$ past,

$$\mathbb{P}(X_2 > t \mid X_1 = t_1, N(t_1) = 1)$$

$$= \mathbb{P}(\underbrace{N(t_1+t) - N(t_1)}_{\sim \mathcal{P}(\lambda t)} = 0)$$

$$= e^{-\lambda t}$$

Variable X_{n+1}

$$\mathbb{P}(X_{n+1} > t \mid X_1 = t_1 , \ldots, X_n = t_n)$$

$$= \mathbb{P}(\text{No arrival on } [\overbrace{t_1 + \cdots + t_n}^{\tau} , \tau + t] \mid X_1 = t_1 , \ldots, X_n = t_n , N(\tau) = n)$$

$\perp\!\!\!\perp$ past

$$= \mathbb{P}(N(\tau + t) - N(\tau) = 0)$$

$$= e^{-\lambda t}$$

$$\Rightarrow \quad X_{n+1} \perp\!\!\!\perp (X_1, \ldots, X_n)$$

$$\boxed{X_{n+1} \sim \mathcal{E}(\lambda)}$$

Proof of Theorem 4 (1)

Variable X_1: We have

$$\mathbf{P}(X_1 > t) = \mathbf{P}(N(t) = 0) = \exp(-\lambda t)$$

Thus

$$X_1 \sim \mathcal{E}(\lambda)$$

Proof of Theorem 4 (2)

Conditioning on X_1: Write

$$\begin{aligned}
&\mathbf{P}\left(X_2 > t \middle| X_1 = t_1\right) \\
=&\mathbf{P}\left(\text{No arrival in } (t_1, t_1 + t] \middle| X_1 = t_1\right) \\
=&\mathbf{P}\left(N(t_1, t_1 + t]) = 0 \middle| N(t_1) = 1,\ X_1 = t_1\right) \\
=&\exp(-\lambda\, t)
\end{aligned}$$

Thus

$$X_2 \sim \mathcal{E}(\lambda), \quad \text{and} \quad X_2 \perp\!\!\!\perp X_1$$

Proof of Theorem 4 (3)

Conditioning on X_n: Write $\tau = \sum_{i=1}^{n} t_i$ and

$$\begin{aligned} & \mathbf{P}\left(X_{n+1} > t \,\middle|\, X_1 = t_1, \ldots, X_n = t_n\right) \\ = & \mathbf{P}\left(\text{No arrival in } (\tau, \tau + t] \,\middle|\, X_1 = t_1, \ldots, X_n = t_n\right) \\ = & \mathbf{P}\left(N(\tau, \tau + t] = 0 \,\middle|\, N(\tau) = 1,\, X_1 = t_1, \ldots, X_n = t_n\right) \\ = & \exp(-\lambda\, t) \end{aligned}$$

Thus

$$X_{n+1} \sim \mathcal{E}(\lambda), \quad \text{and} \quad X_{n+1} \perp\!\!\!\perp (X_1, \ldots, X_n)$$

Another proof of $N(t) \sim \mathcal{P}(\lambda t)$

Strategy:

1. Start from $\{X_k;\ k \geq 1\}$ inter-arrival times
2. Set $T_n = \sum_{k=1}^{n} X_k$ = sum of ⫫ $\mathcal{E}(\lambda)$ r.v. $\Rightarrow T_n \sim \Gamma(\lambda, n)$
3. If X_k's are i.i $\mathcal{E}(\lambda)$ random variables, then $T_n \sim \Gamma(\lambda, n)$
4. Compute

$$
\begin{aligned}
\mathbf{P}\left(N(t) = j\right) &= \mathbf{P}\left(T_j \leq t < T_{j+1}\right) \\
&= \mathbf{P}\left(T_j \leq t\right) - \mathbf{P}\left(T_{j+1} \leq t\right) \\
&\vdots \text{ computations} \\
&= \frac{(\lambda t)^j}{j!} \exp(-\lambda t)
\end{aligned}
$$